Stochastic Calculus

Week 4

Topics:

Ordinary (Newtonian) Calculus

The Riemann-Stieltjes Integral

The Itô Integral – an Example

Construction of the Itô Integral

Itô processes and Itô's lemma

Ordinary (Newtonian) Calculus

Let $F(t)$ be a differentiable function, such that its derivative

$$\frac{dF(t)}{dt} = \lim_{h\to 0} \frac{F(t+h) - F(t)}{h}$$

exists. Then the following three statements are equivalent:

1. $\frac{dF(t)}{dt} = f(t)$ (definition of derivative)
2. $dF(t) = f(t)dt$ (differential equation)
3. $F(t) = F(0) + \int_0^t f(s)ds$ (integral equation)

This gives an easy way to interpret integrals like

$$\int_0^t g(s)dF(s) = \int_0^t g(s)f(s)ds.$$

In stochastic calculus, we want to consider similar expressions with $F(t)$ replaced by a Brownian motion process B_t. However, since Brownian sample paths are not differentiable, the above intuition will not work: there is no process $B_t' = dB_t/dt$, and hence we cannot express $\int_0^t g(s)dB_s$ as $\int_0^t g(s)B_s'ds$.