ECN 510 Exam 2

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1.

1. Since we want a portfolio with minimum risk (standard deviation), we need to combine assets with the smallest standard deviation. Given our assets, these are μ_f and μ_m, the riskless asset and the market asset. Combining these with $x = 0.5$ gives us the expected return and standard deviation of

$$\begin{aligned}
\mu =& x\mu_m + (1-x)\mu_f \\
\mu =& 0.5(0.04) + 0.5(0.12) = 0.08 \\
\sigma^2 =& x^2\sigma_m^2 + (1-x)^2\sigma_f^2 + 2x(1-x)\rho_{mf}\sigma_m\sigma_f \\
& \text{Note:} \sigma_f = 0 \\
\sigma^2 =& x^2\sigma_m^2 \\
\sigma =& \sqrt{x^2\sigma_m^2} \\
\sigma =& \sqrt{0.5^2(0.2)^2} = 0.1
\end{aligned}$$

2. The only possible way to construct a portfolio with an expected return this high is to have it consist solely of asset r_a. Adding any other asset would mean reducing the total μ for the portfolio below 0.16.

3. Yes, it does.

$$\begin{aligned}
\beta_a =& \frac{r_a - r_f}{r_m - r_f} \\
\beta_a =& \frac{0.16 - 0.04}{0.12 - 0.04} \\
\beta_a =& 1.5
\end{aligned}$$

2.

1.

$$\begin{aligned}
\mu_r =& x\mu_a + (1-x)\mu_b \\
\mu_r =& x0.0095 + (1-x)0.019 \\
\sigma_r^2 =& x^2\sigma_a^2 + (1-x)^2\sigma_b^2 + 2x(1-x)\rho_{ab}\sigma_a\sigma_b \\
\sigma_r =& \sqrt{x^2\sigma_a^2 + (1-x)^2\sigma_b^2 + 2x(1-x)\rho_{ab}\sigma_a\sigma_b} \\
\sigma_r =& \sqrt{x^2 0.00586756 + (1-x)^2 0.006561 + x(1-x)0.000094601}
\end{aligned}$$

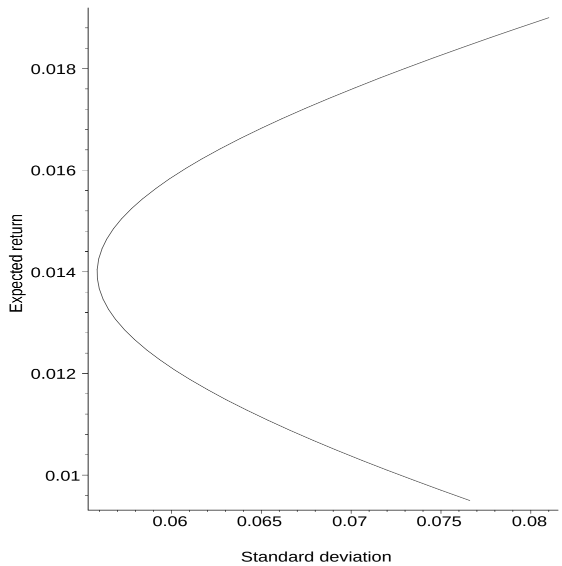

Figure 1: Efficient frontier

x	σ_r^2	μ_r
0.00	0.08100000000	0.0119
0.25	0.06381542691	0.011300
0.50	0.05592231174	0.01070
0.75	0.06103841997	0.010100
1.00	0.07660000000	0.0095

2.

$$\sigma_r^2 = x^2\sigma_a^2 + (1-x)^2\sigma_b^2 + 2x(1-x)\rho_{ab}\sigma_a\sigma_b$$
$$\sigma_r^2 = x^2\sigma_a^2 + \sigma_b^2 x^2 - 2x\sigma_b^2 + \sigma_b^2 + 2x\rho_{ab}\sigma_a\sigma_b - 2x^2\rho\sigma_a\sigma_b$$
$$\sigma_r^{2'} = 2x\sigma_a^2 + 2x\sigma_b^2 - 2\sigma_b^2 + 2\rho_{ab}\sigma_a\sigma_b - 4x\rho_{ab}\sigma_a\sigma_b$$

Setting $\sigma_r^{2'} = 0$:

$$\begin{aligned}
0 =& 2x\sigma_a^2 + 2x\sigma_b^2 - 2\sigma_b^2 + 2\rho_{ab}\sigma_a\sigma_b - 4x\rho_{ab}\sigma_a\sigma_b \\
2\sigma_b^2 - 2\rho_{ab}\sigma_a\sigma_b =& 2x\sigma_a^2 + 2x\sigma_b^2 + 4x\rho_{ab}\sigma_a\sigma_b \\
2\sigma_b^2 - 2\rho_{ab}\sigma_a\sigma_b =& x(2\sigma_a^2 + 2\sigma_b^2 + 4\rho_{ab}\sigma_a\sigma_b) \\
x =& \frac{\sigma_b^2 - \rho_{ab}\sigma_a\sigma_b}{\sigma_a^2 + \sigma_b^2 + 2\rho_{ab}\sigma_a\sigma_b} \\
x =& \frac{(0.081)^2 - (0.0065)(0.0766)(0.081)}{(0.0766)^2 + (0.081)^2 + 2(0.0065)(0.0766)(0.081)} \\
x =& 0.5229551429
\end{aligned}$$

So, we can now calculate μ_r and σ_r:

$$\begin{aligned}
\mu_r =& x\mu_a + x\mu_b = 0.01064490766 \\
\sigma_r =& \sqrt{x^2\sigma_a^2 + (1-x)^2\sigma_b^2 + 2x(1-x)\rho_{ab}\sigma_a\sigma_b} = 0.05583810098
\end{aligned}$$

3. We now create a new portfolio of two assets, the minimum variance portfolio found above and the riskless asset. Our x is

$$\begin{aligned}
x =& \frac{(\mu_a - \mu_f)\sigma_b - (\mu_b - \mu_f)cov(a,b)}{(\mu_a - \mu_f)\sigma_b^2 + (\mu_b - \mu_f)\sigma_a^2 - [(\mu_a - \mu_f) + (\mu_b - \mu_f)]cov(a,b)} \\
& \text{Note: } cov(\mu_a, \mu_b) = \rho_{ab}\sigma_a\sigma_b \\
x =& \frac{(\mu_a - \mu_f)\sigma_b - (\mu_b - \mu_f)\rho_{ab}\sigma_a\sigma_b}{(\mu_a - \mu_f)\sigma_b^2 + (\mu_b - \mu_f)\sigma_a^2 - [(\mu_a - \mu_f) + (\mu_b - \mu_f)]\rho_{ab}\sigma_a\sigma_b} \\
x =& 0.3699764368
\end{aligned}$$

So we get the following for μ and σ:

$$\begin{aligned}
\mu_p =& x\mu_r + (1-x)\mu_f = 0.008222525240 \\
\sigma_p =& \sqrt{x^2\sigma_r^2 + (1-x)^2\sigma_f^2 + 2x(1-x)\rho_{rf}\sigma_r\sigma_f} \\
& \text{Note: } cov(\mu_a, \mu_b) = \rho_{ab}\sigma_a\sigma_b \\
=& \sqrt{x^2\sigma_r^2 + (1-x)^2\sigma_f^2} \\
& \text{Note: } \sigma_f = 0 \\
=& \sqrt{x^2\sigma_r^2} \\
=& 0.08742575376
\end{aligned}$$

We find the CML using values we already have:

$$\begin{aligned}
\mu_p =& r_f + \frac{\mu_r - r_f}{\sigma_r}\sigma_p \\
\mu_p =& 0.0068 + \frac{0.01064490766 - 0.0068}{0.05583810098}\sigma_p \\
=& 0.0068 + 0.06885813795\sigma_p
\end{aligned}$$

4. We can incorporate a utility function into the analysis by finding where it is tangent to the CML. The utility function won't necessarily be tangent to the efficient frontier, as the investor will be combining their bundle of risky assets with the riskless asset. We can find where it is tangent to the CML by putting it in terms of μ, then taking the derivative and setting it equal to the slope of the CML.

$$\begin{aligned}
U =& \mu - 0.2\sigma^2 \\
\mu =& U + 0.2\sigma^2 \\
\mu' =& 0.4\sigma \\
0.06885813795 =& 0.4\sigma \\
\sigma =& \frac{0.06885813795}{0.4} = 0.1721453449
\end{aligned}$$

Now, since we know what σ, we can substitute this back into the CML and find our μ....

$$\begin{aligned}
\mu =& 0.0068 + 0.06885813795(0.1721453449) \\
\mu =& 0.01865360790
\end{aligned}$$

....which we can finally use to easily find our x.

$$\begin{aligned}
\mu =& x\mu_r + (1 - x)\mu_f \\
\mu =& x\mu_r - x\mu_f + \mu_f \\
\mu - \mu_f =& x(\mu_r - \mu_f) \\
x =& \frac{\mu_p - \mu_f}{\mu_r - \mu_f} \\
x =& \frac{0.01865360790 - 0.0068}{0.01064490766 - 0.0068} = 3.08293695147
\end{aligned}$$