Testing for Structural Change of a Cointegrated Regression in Panel Data

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Abstract

This paper proposes a Wald-type test statistic for testing the structural change of a cointegrated regression model in panel data. The asymptotic distribution of the proposed test statistic is a square of a Bessel process as in Delong (1981) and Andrews (1993). We also show that the test considered in this paper has nontrivial local power against a wide class of alternatives. We show that the spurious break may occur when the error term is $I(1)$. Finally, we propose a bootstrap method to test the common change points across groups.

1 Introduction

This paper, along with Emerson and Kao (1999), is the first step in understanding how to test for structural changes of cointegrated regression models in panel data. Emerson and Kao (1999) developed tests for testing for structural break in a time trend regression in panel data. In this paper, we propose a Wald-type test statistic for detecting a break at an unknown date in a cointegrated regression in panel data. We derive the limiting distribution of the proposed test. We also show that the proposed test has non-trivial local power. The limiting distribution of the Wald test is shown to be a similar, though not identical, form of that of the tests in Andrews (1993). The limiting distribution of our test is a square of a Bessel process as in Delong (1981).

Testing for structural changes has been an important research topic in nonstationary time series econometrics. Recent issues of the *Journal of Business and Economic Statistics* and *Journal of Econometrics* are devoted to such studies, e.g., Chu and White (1992); Hansen (1992); Gregory and Hansen (1996); Campos, Ericsson and Hendry (1996). Kao and Ross (1995) extended the dynamic cumulative sum (CUSUM) test of

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Kramer, Ploberger and Alt (1988) to the model where serial correlation is present. Andrews (1993) showed that the supremum of the usual Chow test over all possible break dates converges to the supremum of a tied-down Bessel process. Quintos and Phillips (1993) proposed LM tests for parameter constancy in cointegrated regressions. Quintos (1997) extended the fluctuations test to a nonstationary reduced rank framework and showed the test converges to a sum of tied-down processes. However, none of these papers has looked at the tests in the context of the panel data except Han and Park (1989), Hansen (1999) and Emerson and Kao (1999). Han and Park (1989) proposed a CUSUM and a CUSUM of squares tests for panel data models. Hansen (1999) developed methods for testing the threshold effects in panel data. Emerson and Kao (1999) proposed test statistics for detecting a break at an unknown date in panel data models with time trend. Emerson and Kao derived the limiting distributions of the proposed tests and tabulated the critical values. Asymptotic results were derived $I(0)$, $I(1)$ and nearly $I(1)$ error terms. They also showed that these tests have non-trivial local power only when the error terms are $I(0)$. In a recent paper, though not a panel context, Bai, Lumsdaine and Stock (1998) developed methods for testing and constructing asymptotically valid confidence intervals for the date of a single break in multivariate time series, including $I(0)$, $I(1)$ and deterministically trending regressors. They showed there are substantial gains by using multivariate time series which have a common break date.

The organization of the paper is as follows. Section 2 introduces the model. Section 3 defines the test statistic. The limiting distributions of the proposed test statistics is established. Section 4 establishes the limiting distribution of the proposed test under local alternatives. Section 5 derives the limiting distribution of the proposed test when the error term is $I(1)$. Section 6 suggests a test of testing the common changes across groups in panel data. In Section 7 we summarize the findings. All proofs are in the Appendix.

A word on notation. We write the integral $\int_0^1 W(s)ds$ as $\int W$ when there is no ambiguity over limits. We define $\Omega^{1/2}$ to be any matrix such that $\Omega = \left(\Omega^{1/2}\right)\left(\Omega^{1/2}\right)'$. We use $\|\cdot\|$ to denote the Euclidean norm of a vecor, $\Rightarrow$ to denote weak convergence, $\xrightarrow{p}$ to denote convergence in probability, $[x]$ to denote the largest integer $\leq x$, $I(0)$ and $I(1)$ to signify a time-series that is integrated of order zero and one, respectively, and $BM(\Omega)$ to denote Brownian motion with the covariance matrix Ω.

2 The Model and Assumptions

Consider the following fixed effect panel regression:

$$y_{it} = \alpha_i + x_{it}'\beta_t + u_{it}, \; i = 1, ..., n, \; t = 1, ..., T, \tag{1}$$

where $\{y_{it}\}$ are 1×1, β_t is a $p \times 1$ vector of the slope parameters, $\{\alpha_i\}$ are the intercepts, and $\{u_{it}\}$ are the stationary disturbance terms. We assume that $\{x_{it}\}$ are $p \times 1$ integrated processes of order one for all i, where

$$x_{it} = x_{it-1} + \varepsilon_{it}.$$

Under these specifications, (1) describes a system of cointegrated regressions, i.e., y_{it} is cointegrated with x_{it}. The initialization of this system is $y_{i0} = x_{i0} = O_p(1)$ as $T \to \infty$ for all i. Next, we characterize the innovation vector $w_{it} = \left(u_{it}, \varepsilon_{it}^{'}\right)^{'}$. We assume that w_{it} satisfies the following assumptions.

Assumption 1 *For each i, we assume $\{w_{it}\}$ have means zero and satisfy the invariance principle, i.e.,*

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{[Tr]} w_{it} \Rightarrow B_i(r) = BM(\Omega) \text{ as } T \to \infty \text{ for all } i, \tag{2}$$

where

$$B_i = \begin{bmatrix} B_{ui} \\ B_{\varepsilon i} \end{bmatrix}.$$

Assumption 2 *The $\{w_{it}\}$ are assumed to be independent across i.*

The long-run covariance matrix of $\{w_{it}\}$ is given by

$$\begin{aligned} \Omega &= \sum_{j=-\infty}^{\infty} E\left(w_{ij} w_{i0}^{'}\right) \\ &= \Sigma + \Gamma + \Gamma^{'} \\ &= \begin{bmatrix} \Omega_u & \Omega_{u\varepsilon} \\ \Omega_{\varepsilon u} & \Omega_{\varepsilon} \end{bmatrix}, \end{aligned}$$

where

$$\Gamma = \sum_{j=1}^{\infty} E\left(w_{ij} w_{i0}^{'}\right) = \begin{bmatrix} \Gamma_u & \Gamma_{u\varepsilon} \\ \Gamma_{\varepsilon u} & \Gamma_{\varepsilon} \end{bmatrix} \tag{3}$$

and

$$\Sigma = E\left(w_{i0} w_{i0}^{'}\right) = \begin{bmatrix} \Sigma_u & \Sigma_{u\varepsilon} \\ \Sigma_{\varepsilon u} & \Sigma_{\varepsilon} \end{bmatrix} \tag{4}$$

are partitioned conformably with w_{it}.

Assumption 3 *Ω_{ε} is non-singular, i.e., $\{x_{it}\}$ are not cointegrated.*

Define

$$\Omega_{u.\varepsilon} = \Omega_u - \Omega_{u\varepsilon}\Omega_{\varepsilon}^{-1}\Omega_{\varepsilon u}. \tag{5}$$

Then, B_i can be rewritten as

$$B_i = \begin{bmatrix} B_{ui} \\ B_{\varepsilon i} \end{bmatrix} = \begin{bmatrix} \Omega_{u.\varepsilon}^{1/2} & \Omega_{u\varepsilon}\Omega_{\varepsilon}^{-1/2} \\ 0 & \Omega_{\varepsilon}^{1/2} \end{bmatrix} \begin{bmatrix} V_i \\ W_i \end{bmatrix}, \tag{6}$$

where $\begin{bmatrix} V_i \\ W_i \end{bmatrix} = BM\left(I\right)$ is a standardized Brownian motion. Define the one-sided long-run covariance

$$\begin{aligned} \Delta &= \Sigma + \Gamma \\ &= \sum_{j=0}^{\infty} E\left(w_{ij}w_{i0}^{'}\right) \end{aligned}$$

with

$$\Delta = \begin{bmatrix} \Delta_u & \Delta_{u\varepsilon} \\ \Delta_{\varepsilon u} & \Delta_{\varepsilon} \end{bmatrix}.$$

Here we assume that panels are homogeneous, i.e., the variances are constant across the cross-section units. The problem of interest is to test the changes in the parameter β_t where the change points are unknown. Consider the alternative hypothesis that there is only one change point k, i.e.,

$$H_1 : \beta_t = \begin{cases} \beta_1 & \text{for } t = 1, ..., k \\ \beta_2 & \text{for } t = k+1, ..., T \end{cases}. \tag{7}$$

Under (7), (1) can be written as

$$\begin{aligned} y_{it} &= \alpha_i + x_{it}^{'} I(t \leq k)\beta_1 + x_{it}^{'} I(t > k)\beta_2 + u_{it} \\ &= \alpha_i + x_{it}^{'}\left(k\right)\beta + u_{it} \end{aligned} \tag{8}$$

where $\beta = \left(\beta_1, \beta_2\right)^{'}$, $I(.)$ is the indicator function, and

$$x_{it}(k) = \begin{cases} x_{it}I(t \leq k) \\ x_{it}I(t > k) \end{cases}.$$

Let

$$\begin{aligned} \overline{x}_i(k) &= \frac{1}{T}\sum_{t=1}^{T} x_{it}\left(k\right) \\ &= \begin{cases} \frac{1}{T}\sum_{t=1}^{T} x_{it}I(t \leq k) \\ \frac{1}{T}\sum_{t=1}^{T} x_{it}I(t > k) \end{cases}. \end{aligned}$$

The usual fixed effect representation of (8) is

$$y_{it}^{*}=x_{it}^{*}(k)^{'}\beta+u_{it}^{*}, \tag{9}$$

where

$$y_{it}^{*}=y_{it}-\overline{y}_{i},$$

$$x_{it}^{*}(k)=x_{ii}(k)-\overline{x}_{i}(k),$$

and

$$u_{it}^{*}=u_{it}-\overline{u}_{i}.$$

Let

$$\widehat{u}_{it}^{*}(k)=y_{it}^{*}-x_{t}^{*}(k)^{'}\widehat{\beta}^{*}(k)$$

and

$$S(k)=\sum_{i=1}^{n}\sum_{t=1}^{T}\widehat{u}_{it}^{*}(k)^{2}$$

where $\widehat{\beta}^{*}(k)$ can be obtained by using the FM estimator in (13). Then the least squares estimate of the change point, k, is defined as

$$\widehat{k}=\arg\min_{k} S(k).$$

3 The Wald Test Statistic and Its Limiting Distribution

Define

$$u_{it}^{+}=u_{it}-\Omega_{u\varepsilon}\Omega_{\varepsilon}^{-1}\varepsilon_{it},$$

$$\widehat{u}_{it}^{+}\;=\;u_{it}-\widehat{\Omega}_{u\varepsilon}\widehat{\Omega}_{\varepsilon}^{-1}\varepsilon_{it}, \tag{10}$$

$$y_{it}^{+}\;=\;y_{it}-\Omega_{u\varepsilon}\Omega_{\varepsilon}^{-1}\Delta x_{it}, \tag{11}$$

and

$$\widehat{y}_{it}^{+}\;=\;y_{it}-\widehat{\Omega}_{u\varepsilon}\widehat{\Omega}_{\varepsilon}^{-1}\Delta x_{it}. \tag{12}$$

Note that

$$\begin{bmatrix} u_{it}^{+} \\ \varepsilon_{it} \end{bmatrix}=\begin{bmatrix} 1 & -\Omega_{u\varepsilon}\Omega_{\varepsilon}^{-1} \\ 0 & \mathbf{I}_{p} \end{bmatrix}\begin{bmatrix} u_{it} \\ \varepsilon_{it} \end{bmatrix},$$

which has the long-run covariance matrix

$$\left[\begin{array}{cc} \Omega_{u.\varepsilon} & 0 \\ 0 & \Omega_{\varepsilon} \end{array}\right],$$

where $\mathbf{I}_p$ is a $p \times p$ identity matrix. The endogeneity correction is achieved by modifying the variable y_{it} in (1) with the transformation

$$\begin{aligned} \widehat{y}_{it}^{+} &= y_{it} - \widehat{\Omega}_{u\varepsilon}\widehat{\Omega}_{\varepsilon}^{-1}\Delta x_{it} \\ &= \alpha_i + x_{it}^{'}\beta + u_{it} - \widehat{\Omega}_{u\varepsilon}\widehat{\Omega}_{\varepsilon}^{-1}\Delta x_{it}. \end{aligned}$$

The serial correlation correction term has the form

$$\begin{aligned} \widehat{\Delta}_{\varepsilon u}^{+} &= \left(\begin{array}{cc} \widehat{\Delta}_{\varepsilon u} & \widehat{\Delta}_{\varepsilon} \end{array} \right) \left(\begin{array}{c} 1 \\ -\widehat{\Omega}_{\varepsilon}^{-1}\widehat{\Omega}_{\varepsilon u} \end{array} \right) \\ &= \widehat{\Delta}_{\varepsilon u} - \widehat{\Delta}_{\varepsilon}\widehat{\Omega}_{\varepsilon}^{-1}\widehat{\Omega}_{\varepsilon u}, \end{aligned}$$

where $\widehat{\Delta}_{\varepsilon u}$ and $\widehat{\Delta}_{\varepsilon}$ are kernel estimates of $\Delta_{\varepsilon u}$ and Δ_{ε}. Therefore, the FM estimator under H_0 is

$$\widehat{\beta} = \left[\sum_{i=1}^{n} \sum_{t=1}^{T} (x_{it} - \overline{x}_i)(x_{it} - \overline{x}_i)^{'} \right]^{-1} \left[\sum_{i=1}^{n} \left(\sum_{t=1}^{T} (x_{it} - \overline{x}_i)\, \widehat{y}_{it}^{+} - T\widehat{\Delta}_{\varepsilon u}^{+} \right) \right]. \tag{13}$$

The limiting distribution of the FM estimator under H_0 has been developed by Kao and Chiang (1999) and Pedroni (1996). All limits in Lemma 1-2, Theorems 1-3 and Corollary 1 are taken as $T \to \infty$ followed by $n \to \infty$ sequentially.

Lemma 1 *Kao and Chiang (1999). Suppose Assumptions 1-3 and under H_0 hold. Then*

$$\sqrt{n}T\left(\widehat{\beta} - \beta\right) \Rightarrow \mathrm{N}\left(0, 6\Omega_{\varepsilon}^{-1}\Omega_{u.\varepsilon}\right).$$

We are interested in testing for the stability of β in (1), i.e., we test $H_0 : \beta_1 = \beta_2$ against $H_1 : \beta_1 \neq \beta_2$. Let $W(k)$ be the Wald statistic:

$$\mathrm{W}(k) = \frac{1}{\widehat{\Omega}_{u.\varepsilon}} \left(\widehat{\beta}_{1k} - \widehat{\beta}_{2k}\right)^{'} \left[\begin{array}{c} \left(\sum_{i=1}^{n}\sum_{t=1}^{k}(x_{it} - \overline{x}_{1ik})^2\right)^{-1} \\ + \left(\sum_{i=1}^{n}\sum_{t=k+1}^{T}(x_{it} - \overline{x}_{2ik})^2\right)^{-1} \end{array} \right]^{-1} \left(\widehat{\beta}_{1k} - \widehat{\beta}_{2k}\right), \tag{14}$$

where $\widehat{\Omega}_{u.\varepsilon}$ is a consistent estimator $\Omega_{u.\varepsilon}$ under H_0 (e.g., Kao and Chiang, 1999).

Assumption 4 $\frac{k}{T} \to r$ *as T and $k \to \infty$.*

Then we have the following theorem:

Theorem 1 *Suppose the conditions of Assumptions 1-4 and the null hypothesis, H_0, hold. Then*

$$\mathrm{W}(k) \Rightarrow Q_p(r)$$

uniformly in r, where $Q_p(r) = \frac{1}{r^2+(1-r)^2}\left[B((1-r)^2) - B\left(r^2\right)\right]'\left[B((1-r)^2) - B\left(r^2\right)\right].$

Remark 1 1. *Note that the limiting distribution of $\mathrm{W}(k)$ is nuisance parameter free and depends only on the number of regressors, p.*

2. *Note that $B((1-r)^2) - B\left(r^2\right)$ has variance $(1-r)^2 - r^2$. Then $\left\|B((1-r)^2) - B\left(r^2\right)\right\|$ is a Bessel process of order p. Therefore we can write*

$$\frac{1}{r^2+(1-r)^2}\left[B((1-r)^2) - B\left(r^2\right)\right]'\left[B((1-r)^2) - B\left(r^2\right)\right]$$

as

$$\frac{(1-r)^2 - r^2}{(1-r)^2 + r^2}\frac{\left[B((1-r)^2) - B\left(r^2\right)\right]'\left[B((1-r)^2) - B\left(r^2\right)\right]}{(1-r)^2 - r^2}$$

where

$$\frac{\left[B((1-r)^2) - B\left(r^2\right)\right]'\left[B((1-r)^2) - B\left(r^2\right)\right]}{(1-r)^2 - r^2}$$

is a square of a standardized Bessel process. Let $s = (1-r)^2 - r^2$. Then

$$\begin{aligned} &\frac{\left[B((1-r)^2) - B\left(r^2\right)\right]'\left[B((1-r)^2) - B\left(r^2\right)\right]}{(1-r)^2 - r^2} \\ = &\frac{BM(s)'BM(s)}{s}, \end{aligned}$$

where $BM(s)$ denotes a p-vector of independent Brownian processes on $[0,\infty]$. For a fixed r, $\frac{BM(s)'BM(s)}{s}$ has a chi-squared distribution with p degrees of freedom. However, r cannot be $\frac{1}{2}$ since s will be zero when $r = \frac{1}{2}$.

3. *The limiting distribution of $\frac{(1-r)^2+r^2}{(1-r)^2-r^2}\mathrm{W}(k)$, has the same form as in Delong (1981) and Andrews (1993). The result is in contrast to that of Hansen (1992) and Quintos and Phillips (1993), who consider the tests for testing for structural change with $I(1)$ regressors in pure time series, in which the test statistics depend on whether the regressors are $I(0)$ or $I(1)$.*

4. *The results in Theorem 1 and the rest of the paper will continue to hold if we replace the FM estimator by the dynamic OLS (DOLS) of Kao and Chiang (1999) since FM and DOLS estimators are asymptotically equivalent.*

5. *The results in Theorem 1 and the rest of the paper will continue to hold if we include a time trend or other deterministic component in (1).*

Next we consider three statistics: $sup\mathrm{W}(k)$, $Mean\mathrm{W}(k)$, and $Exp\mathrm{W}(k)$, where

$$sup\mathrm{W}(k) = \sup_{[Tr^*]\leq k\leq T-[Tr^*]} \mathrm{W}(k),$$

$$Mean\mathrm{W}(k) = \frac{1}{T}\sum_{k=[Tr^*]}^{T-[Tr^*]} \mathrm{W}(k),$$

and

$$Exp\mathrm{W}(k) = \log\left(\frac{1}{T}\sum_{k=[Tr^*]}^{T-[Tr^*]} \exp\left(\frac{1}{2}\mathrm{W}(k)\right)\right).$$

Using the continuous mapping theorem we then have the following corollary:

Corollary 2 *Suppose the assumptions in Theorem 1 hold and under* H_0 :

1. $sup\mathrm{W}(k) \Rightarrow \sup_{r^*\leq r\leq 1-r^*} Q_p(r),$
2. $Mean\mathrm{W}(k) \Rightarrow \int_{r^*}^{1-r^*} Q_p(r)dr,$
3. $Exp\mathrm{W}(k) \Rightarrow \log\left(\int_{r^*}^{1-r^*} \exp\left(\frac{1}{2}Q_p(r)\right)dr\right).$

Remark 2 1. *Note the asymptotic null distributions of* $sup\mathrm{W}(k)$, $Mean\mathrm{W}(k)$, *and* $Exp\mathrm{W}(k)$ *are the same given in Delong (1981) and Andrews and Ploberger (1994).*

Critical values c_α for the test statistics $sup\mathrm{W}(k)$, $Mean\mathrm{W}(k)$, and $Exp\mathrm{W}(k)$ are provided in Table 1.By definition, c_α, satisfies P

4 Local Asymptotic Power

It is important to know if the Wald test has good power properties under general alternative hypotheses. To examine asymptotic local power we consider the following local alternative:

$$\beta_t^{(T)} = \beta + \frac{1}{T}g\left(\frac{t}{T}\right), \tag{15}$$

where g is a $p\times 1$ arbitrary function defined on unit interval. Define

$$y_{it}^{(T)} = \alpha_i + x_{it}^{'}\beta_t^{(T)} + u_{it}$$

and let $\widehat{\beta}_{1k}^{(T)}$ and $\widehat{\beta}_{2k}^{(T)}$ be the FM estimators under the local alternative (15). Similarly, let

$$\mathrm{W}^{(T)}(k) = \frac{1}{\widetilde{\Omega}_{u.\varepsilon}} \left(\widehat{\beta}_{1k}^{(T)} - \widehat{\beta}_{2k}^{(T)}\right)^{'} \left[\begin{array}{c} \left(\sum_{i=1}^{n}\sum_{t=1}^{k}(x_{it} - \overline{x}_{1ik})^2\right)^{-1} \\ + \left(\sum_{i=1}^{n}\sum_{t=k+1}^{T}(x_{it} - \overline{x}_{2ik})^2\right)^{-1} \end{array} \right]^{-1} \left(\widehat{\beta}_{1k}^{(T)} - \widehat{\beta}_{2k}^{(T)}\right)$$

be the corresponding test statistic under local alternative, where $\widetilde{\Omega}_{u.\varepsilon}$ is the estimate of $\Omega_{u.\varepsilon}$ under local alternatives.

Theorem 2 *Suppose Assumptions 1-4 hold. Under the local alternatives (15), then*

$$W^{(T)}(k) \Rightarrow Q_p(r) + O_p(1)$$

uniformly in r.

Remark 3 *Theorem 2 indicates that Wald test in (14) has nontrivial local power irrespective of the particular type of the structural change.*

5 Spurious Break

We now extend the results of the previous sections to the case where the error term in (1) is $I(1)$. Consider

$$y_{it} = \alpha_i + x_{it}^{'}\beta + e_{it}, \tag{16}$$

$$x_{it} = x_{it-1} + \varepsilon_{it},$$

where e_{it} is $I(1)$.

Assumption 5 *Suppose that $w_{it} = (u_{it}, \varepsilon_{it})^{'}$ is a bivariate process with zero mean vector and the long-run covariance matrix of w_{it}:*

$$\Omega = \Sigma + \Gamma + \Gamma^{'} = \begin{bmatrix} \Omega_u & \Omega_{u\varepsilon} \\ \Omega_{\varepsilon u} & \Omega_{\varepsilon} \end{bmatrix},$$

where

$$\Gamma = \begin{bmatrix} \Gamma_u & \Gamma_{u\varepsilon} \\ \Gamma_{\varepsilon u} & \Gamma_{\varepsilon} \end{bmatrix}$$

and

$$\Sigma = \begin{bmatrix} \Sigma_u & \Sigma_{u\varepsilon} \\ \Sigma_{\varepsilon u} & \Sigma_{\varepsilon} \end{bmatrix}.$$

Let $y_{it} = \sum_{s=1}^{t} u_{is}$ and $x_{it} = \sum_{s=1}^{t} \varepsilon_{is}$ in which $u_{i0} = \varepsilon_{i0} = O_p(1)$. Suppose y_{it} and x_{it}, are incorrectly estimated by least squares for all i using panel data; the spurious OLS regression model. The OLS estimator of β is

$$\widehat{\beta} = \left[\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)\left(x_{it}-\overline{x}_i\right)'\right]^{-1}\left[\sum_{i=1}^{n}\left(\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)y_{it}\right)\right]. \tag{17}$$

Let $\Omega_{u.\varepsilon} = \Omega_u - \Omega_{u\varepsilon}\Omega_{\varepsilon}^{-1}\Omega_{\varepsilon u}$ and $\beta = \Omega_{\varepsilon}^{-1}\Omega_{u\varepsilon}$, then we have following lemma from Kao (1999):

Lemma 3 *Kao (1999). Under H_0 and assume that Assumptions 2-5 are satisfied and e_{it} is $I(1)$ in (16), then*

1. $\widehat{\beta} \xrightarrow{p} \beta$,
2. $\sqrt{n}\left(\widehat{\beta} - \beta\right) \Rightarrow \mathrm{N}\left(0, \frac{2}{5}\Omega_{\varepsilon}^{-1}\Omega_{u.\varepsilon}\right)$.

Let $W(k)$ be the Wald statistic:

$$\mathrm{W}(k) = \frac{1}{\widehat{\Omega}_{u.\varepsilon}}\left(\widehat{\beta}_{1k} - \widehat{\beta}_{2k}\right)'\left[\begin{array}{c}\left(\sum_{i=1}^{n}\sum_{t=1}^{k}(x_{it}-\overline{x}_{1ik})^2\right)^{-1} \\ +\left(\sum_{i=1}^{n}\sum_{t=k+1}^{T}(x_{it}-\overline{x}_{2ik})^2\right)^{-1}\end{array}\right]^{-1}\left(\widehat{\beta}_{1k} - \widehat{\beta}_{2k}\right). \tag{18}$$

The limiting distribution under H_0 of $\mathrm{W}(k)$ is given in the next theorem:

Theorem 3 *Suppose (16) is true. Under H_0 and given assumptions 2-5,*

$$\frac{1}{T^2}\mathrm{W}(k) \Rightarrow \frac{1}{15}Q_p(r)$$

uniformly in r.

Remark 4

1. *Observe that the Wald statistic, $\mathrm{W}(k)$, will be unbounded asymptotically under H_0 and when the error term is $I(1)$ in (16), i.e., the null of no break will be rejected with probability one asymptotically. This is called the spurious break. The spurious break phenomenon in the pure time series has been discussed by Nunes, Kuan, and Newbold (1995) and Bai (1998). However, $\frac{1}{T^2}\mathrm{W}(k)$ can be used to test for the break when the error term is $I(1)$.*
2. *Suppose β differs across i in (1) and we use the OLS in (17) to estimate this heterogeneous panel cointegration model. Then this misspecification of the heterogeneous panel cointegration will cause the spurious break.*

6 Testing for the Common Change Point Across Groups

The assumption of having the common changes across i in (1) is rather restrictive. We now consider a test that tests for the common change point across groups. Suppose there are two subgroups in the cross-sectional units and we generalize (8) as

$$y_{it} = \begin{cases} \alpha_{ai} & + \quad x_{it}^{'} I(t \le k_a)\beta_{1a} + x_{it}^{'} I(t > k_a)\beta_{2a} \quad + \quad u_{it}, \quad \text{for } i \in I_a \\ \alpha_{bi} & + \quad x_{it}^{'} I(t \le k_b)\beta_{1b} + x_{it}^{'} I(t > k_b)\beta_{2b} \quad + \quad u_{it}, \quad \text{for } i \in I_a \end{cases}, \tag{19}$$

where I_a and I_b are index sets corresponding to subgroups in the cross-section units. Let $n_a = \#(I_a)$ and $n_b = \#(I_b)$. The fixed effect representation of (19) is

$$y_{it}^{*} = \begin{cases} x_{it}^{*}(k_a)^{'}\beta_a \quad + \quad u_{it}^{*}, & \text{for } i \in I_a \\ x_{it}^{*}(k_b)^{'}\beta_b \quad + \quad u_{it}^{*}, & \text{for } i \in I_a \end{cases}, \tag{20}$$

where

$$y_{it}^{*} = y_{it} - \overline{y}_i,$$

$$x_{it}^{*}(k_j) = x_{ii}(k_j) - \overline{x}_i(k_j),$$

$$\overline{x}_i(k_j) = \begin{cases} \frac{1}{T}\sum_{t=1}^{T} x_{it} I(t \le k_j) \\ \frac{1}{T}\sum_{t=1}^{T} x_{it} I(t > k_j) \end{cases}, \ j = a, b,$$

and

$$u_{it}^{*} = u_{it} - \overline{u}_i.$$

For any given k_a and k_b, the slope coefficients β_a and β_b can be estimated by FM estimator in (13). Once $\widehat{k}_a$ and $\widehat{k}_b$ are obtained, the slope coefficients are $\widehat{\beta}_a = \widehat{\beta}_a(k_a)$ and $\widehat{\beta}_b = \widehat{\beta}_a(k_a)$. The regression residual is

$$\widehat{u}_{itj}^{*} = \begin{cases} y_{it}^{*} - x_{it}^{*}(k_a)^{'}\widehat{\beta}_a, & \text{for } i \in I_a \\ y_{it}^{*} - x_{it}^{*}(k_b)^{'}\widehat{\beta}_b. & \text{for } i \in I_a \end{cases}$$

and the sum of squared errors is

$$S_j\left(k_j\right) = \sum_{i=1}^{n_j}\sum_{t=1}^{T}\left(\widehat{u}_{itj}^{*}\right)^2.$$

The least squares of k_j $(j = a, b)$ is defined as

$$\widehat{k}_j = \arg\min_{k_j} S_j(k_j).$$

We are interested in testing

$$H_0 : k_a = k_b = k. \tag{21}$$

Under the null hypothesis of common change points, the model is

$$y_{it} = \begin{cases} \alpha_{ai} & + & x_{it}^{'} I(t \leq k)\beta_{1a} + x_{it}^{'} I(t > k)\beta_{2a} & + & u_{it}, & \text{for } i \in I_a \\ \alpha_{bi} & + & x_{it}^{'} I(t \leq k)\beta_{1b} + x_{it}^{'} I(t > k)\beta_{2b} & + & u_{it}, & \text{for } i \in I_a \end{cases},$$

Under the null hypothesis, the fixed effect representation is

$$y_{it}^{*} = \begin{cases} x_{it}^{*}(k)^{'}\beta_a & + & u_{it}^{*}, & \text{for } i \in I_a \\ x_{it}^{*}(k)^{'}\beta_b & + & u_{it}^{*}, & \text{for } i \in I_a \end{cases}. \tag{22}$$

The regression parameters β_a and β_b are estimated by FM, yielding $\widehat{\beta}_a(k)$ and $\widehat{\beta}_b(k)$. The least squares estimate of k is

$$\widetilde{k} = \arg\min_{k} \sum_{j} S_j(k), \ \ j = a, b.$$

Under the null hypothesis, the sum of squared errors is

$$S_0 = \sum_j S_j(\widehat{k})$$

The F test of H_0 is based on

$$\begin{aligned} F &= \frac{(S_0 - S_1)}{S_1/d_1} \\ &= \frac{\left(\sum_j \sum_{i=1}^{n_j} \sum_{t=1}^{T} \widehat{u}_{itj}^{*}\left(\widetilde{k}\right)^2 - \sum_j \sum_{i=1}^{n_j} \sum_{t=1}^{T} \widehat{u}_{itj}^{*}\left(\widehat{k}_j\right)^2\right)/d_0}{\sum_j S_j(k_j)/d_1} \\ &= \frac{\sum_j \sum_{i=1}^{n_j} \sum_{t=1}^{T} \left[\widehat{u}_{itj}^{*}\left(\widetilde{k}\right)^2 - \widehat{u}_{itj}^{*}\left(\widehat{k}_j\right)^2\right]}{\sum_j S_j(k_j)/d_1}, \end{aligned} \tag{23}$$

where $d_1 = n(T-1)$, and

$$S_1 = \sum_j S_j(k_j).$$

The asymptotic distribution of F in (23) can be obtained by using a bootstrap procedure:

1. Estimate (20) and (22) by FM. Calculate the F statistic for testing the null $H_0 : k_a = k_b = k$. Compute the regression residuals $\widehat{u}_{itj}^{*}$ and $\triangle x_{it}^{*}$ under H_0.

2. Draw n integers $j_{(1)}, ..., j_{(n)}$ with replacement from the set of integers $1, ..., n$.

3. Let $\widehat{u}_{it}^{*} = (\widehat{u}_{ita}^{*}, \widehat{u}_{itb}^{*})^{'}$ and $\widehat{w}_{it} = (\widehat{u}_{it}^{*}, \triangle x_{it}^{*})^{'}$. For $i = 1, ..., n$, draw $\{w_{it}^{*}\}_{t=1}^{T}$ randomly with replacement from $\left\{\{\widehat{w}_{j_{(i)}t}\}_{t=1}^{T}\right\}_{i=1}^{n}$. We may need bootstrap $\{\widehat{w}_{j_{(i)}t}\}_{t=1}^{T}$ by the recursive or block methods to preserve the serial correlation and endogeneity. For example:

(a) Forming moving block pairs

$$\left(\widehat{w}_{j_{(i)}t}, ..., \widehat{w}_{j_{(i)},t+b-1}\right)$$

from t to $t+b-1$.

(b) Draw blocks $\left(w^*_{it}, ..., w^*_{i,t+b-1}\right)$ randomly with replacement from the original blocks.

4. Randomly divide $\left\{\{w^*_{it}\}_{i=1}^{n}\right\}_{t=1}^{T}$ into two groups $\left\{\{w^*_{it}\}_{i=1}^{n^*_a}\right\}_{t=1}^{T}$ and $\left\{\{w^*_{it}\}_{i=1}^{n^*_b}\right\}_{t=1}^{T}$ where n^*_a and n^*_b are the number of observations .

5. Use $\left\{\{w^*_{it}\}_{i=1}^{n^*_a}\right\}_{t=1}^{T}$ and $\left\{\{w^*_{it}\}_{i=1}^{n^*_b}\right\}_{t=1}^{T}$ to create a bootstrap sample under H_0.

6. Using the bootsrap sample estimate the model under the null and alternative. Calculate the bootstrap value of the F test, F^*, in (23).

7. Repeat steps 2 - 5 B times. The the asymptotic p-value for F test can be calculated by looking at the percentage of draws for which the simulated F statistic exceeds the actual F in Step 1. Then we obtain the bootstrap p-value under H$_o$:

$$p\text{-}value = \frac{1}{B}\sum_{j=1}^{B} I\left(F^*_j > F\right).$$

The null of having common change points will be rejected if the p-value is smaller than a desired value (say 5%).

7 Conclusion

This paper proposes a Wald-type test statistic for testing the structural change of a cointegrated regression model in panel data. The asymptotic distribution of the proposed test statistic is standard and free of nuisance parameters, i.e., it is a square of a Bessel process as in Delong (1981) and Andrew (1993). We also show that the test considered in this paper has nontrivial local power against a wide class of alternatives. Finally, we propose a bootstrap method to test the common change points across groups.

Appendix

A Proof of Theorem 1

Proof. Under H_0 we know

$$\begin{aligned}
& \sqrt{nT}\left(\widehat{\beta}_{1k}-\beta\right) \\
= & \left[\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^2}\sum_{t=1}^{k}\left(x_{it}-\overline{x}_{1i}\right)\left(x_{it}-\overline{x}_{1i}\right)^{'}\right]\sqrt{n}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{k}\left[\left(x_{it}-\overline{x}_{1i}\right)\widehat{u}_{it}^{+}-\widehat{\Delta}_{\varepsilon u}^{+}\right] \\
= & \left[\frac{1}{n}\sum_{i=1}^{n}\zeta_{2iT}\right]^{-1}\left[\sqrt{n}\frac{1}{n}\sum_{i=1}^{n}\zeta_{1iT}\right] \\
= & \left[\xi_{2nT}\right]^{-1}\sqrt{n}\xi_{1nT}.
\end{aligned}$$

where $\zeta_{2iT} = \frac{1}{T^2}\sum_{t=1}^{k}\left(x_{it}-\overline{x}_{1i}\right)\left(x_{it}-\overline{x}_{i}\right)^{'}$, $\xi_{2NT} = \frac{1}{n}\sum_{i=1}^{n}\zeta_{2iT}$, $\zeta_{1iT} = \frac{1}{T}\sum_{t=1}^{k}\left[\left(x_{it}-\overline{x}_{i}\right)\widehat{u}_{it}^{+}-\widehat{\Delta}_{\varepsilon u}^{+}\right]$, and $\xi_{1nT} = \frac{1}{n}\sum_{i=1}^{n}\zeta_{1iT}$.

Let $w_{it}^{+} = \begin{pmatrix} u_{it}^{+} & \varepsilon_{it}^{'} \end{pmatrix}^{'}$ and we have

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{[Tr]} w_{it}^{+} \Rightarrow \begin{bmatrix} B_{ui}^{+} \\ B_{\varepsilon i} \end{bmatrix} = BM\left(\Omega^{+}\right) \text{ as } T\rightarrow\infty,$$

where

$$\Omega^{+} = \begin{bmatrix} \Omega_{u.\varepsilon} & 0 \\ 0 & \Omega_{\varepsilon} \end{bmatrix} = \Sigma^{+}+\Gamma^{+}+\Gamma^{+'}$$

and

$$\begin{bmatrix} B_{ui}^{+} \\ B_{\varepsilon i} \end{bmatrix} = \begin{bmatrix} I & -\Omega_{u\varepsilon}\Omega_{\varepsilon}^{-1} \\ 0 & I \end{bmatrix}\begin{bmatrix} B_{ui} \\ B_{\varepsilon i} \end{bmatrix}.$$

Let

$$\Delta^{+} = \Sigma^{+}+\Gamma^{+}$$

and

$$\zeta_{1iT}^{+} = \frac{1}{T}\sum_{t=1}^{k}\left(x_{it}-\overline{x}_{1i}\right)\widehat{u}_{it}^{+}.$$

From Kao and Chiang (1999) and the consistency of $\widehat{\Omega}_{u\varepsilon}$ and $\widehat{\Omega}_{\varepsilon}^{-1}$ we note that

$$\begin{aligned}
\zeta_{1iT}^{+} & = \frac{1}{T}\sum_{t=1}^{k}\left(x_{it}-\overline{x}_{1i}\right)\widehat{u}_{it}^{+} \\
& \Rightarrow \int_{0}^{r}\widetilde{B}_{\varepsilon i}dB_{ui}^{+}+r\Delta_{\varepsilon u}^{+},
\end{aligned}$$

for all i, where

$$\begin{aligned}
\Delta_{\varepsilon u}^{+} &= \begin{pmatrix} \Delta_{\varepsilon u} & \Delta_{\varepsilon} \end{pmatrix} \begin{pmatrix} 1 \\ -\Omega_{\varepsilon}^{-1}\Omega_{\varepsilon u} \end{pmatrix} \\
&= \Delta_{\varepsilon u} - \Delta_{\varepsilon}\Omega_{\varepsilon}^{-1}\Omega_{\varepsilon u},
\end{aligned}$$

as $T \to \infty$. It follows in terms of standard Wiener processes that

$$\zeta_{1iT}^{+} \Rightarrow \Omega_{\varepsilon}^{1/2}\left(\int_0^r \widetilde{W}_i dV_i\right)\Omega_{u.\varepsilon}^{1/2} + r\Delta_{\varepsilon u}^{+}.$$

Now let

$$\zeta_{1iT} = \zeta_{1iT}^{+} - k\widehat{\Delta}_{\varepsilon u}^{+}.$$

Clearly, we know that

$$\zeta_{1iT} \Rightarrow \Omega_{\varepsilon}^{1/2}\left(\int_0^r \widetilde{W}_i dV_i\right)\Omega_{u.\varepsilon}^{1/2} = \zeta_{1i},$$
$$E\left[\zeta_{1i}\right] = 0,$$

and

$$Var\left[\zeta_{1i}\right] = \frac{r^2}{6}\Omega_{u.\varepsilon}\Omega_{\varepsilon}.$$

Also

$$\begin{aligned}
\zeta_{2iT} &= \frac{1}{T^2}\sum_{t=1}^{k}\left(x_{it} - \overline{x}_{1i}\right)\left(x_{it} - \overline{x}_{1i}\right)' \\
&\Rightarrow \Omega_{\varepsilon}^{1/2}\left(\int_0^r \widetilde{W}_i\widetilde{W}_i'\right)\Omega_{\varepsilon}^{1/2} = \zeta_{2i}
\end{aligned}$$

as $T \to \infty$ for all i, where

$$\widetilde{W}_i = W_i - \frac{1}{r}\int_0^r W_i,$$

as $T \to \infty$ for all i, where $\triangle_{\varepsilon u} = \Sigma_{\varepsilon u} + \Gamma_{\varepsilon u}$. It can be shown that

$$E\left[\int_0^r \widetilde{W}_i\widetilde{W}_i'\right] = \frac{r^2}{6}\mathbf{I}_p,$$

where $\mathbf{I}_p$ is a $p \times p$ identity matrix. It then follows that

$$\begin{aligned}
E\left[\zeta_{2i}\right] &= \frac{r^2}{6}\Omega_{\varepsilon}^{1/2}\mathbf{I}_p\Omega_{\varepsilon}^{1/2} \\
&= \frac{r^2}{6}\Omega_{\varepsilon}.
\end{aligned}$$

Then,

$$\sqrt{n}T\left(\widehat{\beta}_{1k} - \beta\right) \Rightarrow \left[\frac{1}{n}\sum_{i=1}^{n}\Omega_{\varepsilon}^{1/2}\left(\int_0^r \widetilde{W}_i\widetilde{W}_i'\right)\Omega_{\varepsilon}^{1/2}\right]^{-1}\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Omega_{\varepsilon}^{1/2}\left(\int_0^r \widetilde{W}_i dV_i\right)\Omega_{u.\varepsilon}^{1/2}\right]$$

uniformly in r for all fixed n as $T \to \infty$. Thus, applying the multivariate Lindeberg-Levy central limit theorem to $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Omega_{\varepsilon}^{1/2} \left(\int_0^r \widetilde{W}_i dV_i \right) \Omega_{u.\varepsilon}^{1/2}$ to get

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Omega_{\varepsilon}^{1/2} \left(\int_0^r \widetilde{W}_i dV_i \right) \Omega_{u.\varepsilon}^{1/2} \Rightarrow \mathrm{N}\left(0, \frac{r^2}{6} \Omega_{u.\varepsilon} \Omega_{\varepsilon}\right)$$

uniformly in r and combining this with

$$\frac{1}{n} \sum_{i=1}^{n} \Omega_{\varepsilon}^{1/2} \left(\int_0^r \widetilde{W}_i \widetilde{W}_i^{\prime} \right) \Omega_{\varepsilon}^{1/2} \xrightarrow{p} \frac{r^2}{6} \Omega_{\varepsilon},$$

we have

$$\begin{aligned} & \left[\frac{1}{n} \sum_{i=1}^{n} \Omega_{\varepsilon}^{1/2} \left(\int_0^r \widetilde{W}_i \widetilde{W}_i^{\prime} \right) \Omega_{\varepsilon}^{1/2} \right]^{-1} \left[\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Omega_{\varepsilon}^{1/2} \left(\int_0^r \widetilde{W}_i dV_i \right) \Omega_{u.\varepsilon}^{1/2} \right] \\ \Rightarrow \quad & \left[\frac{r^2}{6} \Omega_{\varepsilon} \right]^{-1} \mathrm{N}\left(0, \frac{r^2}{6} \Omega_{u.\varepsilon} \Omega_{\varepsilon}\right) = \frac{\sqrt{6}}{r^2} C B(r^2), \end{aligned}$$

uniformly in r as $n \to \infty$, where $C = \left(\Omega_{\varepsilon}^{-1} \Omega_{u.\varepsilon}\right)^{1/2}$, $B(r)$ is a $p \times 1$ vector of independent Brownian motion. Then, by the sequential limit theory we have

$$\sqrt{n} T \left(\widehat{\beta}_{1k} - \beta \right) \Rightarrow \frac{\sqrt{6}}{r^2} C B(r^2),$$

uniformly in r. Similarly,

$$\frac{1}{T^2} \sum_{t=k+1}^{T} \left(x_{it} - \overline{x}_{2i}\right) \left(x_{it} - \overline{x}_{2i}\right)^{\prime} \xrightarrow{p} \frac{(1-r)^2}{6} \Omega_{\varepsilon},$$

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{T} \sum_{t=k+1}^{T} \left[\left(x_{it} - \overline{x}_{2i}\right) \widehat{u}_{it}^{+} - \widehat{\Delta}_{\varepsilon u}^{+} \right] \Rightarrow \mathrm{N}\left(0, \frac{(1-r)^2}{6} \Omega_{u.\varepsilon} \Omega_{\varepsilon}\right)$$

and

$$\begin{aligned} \sqrt{n} T \left(\widehat{\beta}_{2k} - \beta \right) & \Rightarrow \frac{6}{(1-r)^2} \Omega_{\varepsilon}^{-1} \mathrm{N}\left(0, \frac{(1-r)^2}{6} \Omega_{u.\varepsilon} \Omega_{\varepsilon}\right) \\ & = \frac{\sqrt{6}}{(1-r)^2} \mathrm{N}\left(0, (1-r)^2 \Omega_{\varepsilon}^{-1} \Omega_{u.\varepsilon}\right) \\ & = \frac{\sqrt{6}}{(1-r)^2} C B((1-r)^2) \end{aligned}$$

uniformly in r using

$$\frac{1}{T^2} \sum_{t=k+1}^{T} \left(x_{it} - \overline{x}_{2i}\right) \left(x_{it} - \overline{x}_{2i}\right)^{\prime} \Rightarrow \Omega_{\varepsilon}^{1/2} \left(\int_r^1 \widehat{W}_i \widehat{W}_i^{\prime} \right) \Omega_{\varepsilon}^{1/2\prime},$$

$$\frac{1}{T}\sum_{t=k+1}^{T}\left[\left(x_{it}-\overline{x}_{2i}\right)\widehat{u}_{it}^{+}-\widehat{\Delta}_{\varepsilon u}^{+}\right]\Rightarrow\Omega_{\varepsilon}^{1/2}\left(\int_{r}^{1}\widehat{W}_{i}dV_{i}\right)\Omega_{u.\varepsilon}^{1/2},$$

$$\begin{aligned}
& E\left[\int_{r}^{1}\widehat{W}_{i}\widehat{W}_{i}^{'}\right] \\
= \;& E\left[\int_{r}^{1}W_{i}W_{i}^{'}-\frac{1}{1-r}\int_{r}^{1}W_{i}\int_{r}^{1}W_{i}^{'}\right] \\
= \;& \int_{r}^{1}\left(E\left[W_{i}W_{i}^{'}\right]-\frac{1}{1-r}E\left[\int_{r}^{1}W_{i}\int_{r}^{1}W_{i}^{'}\right]\right) \\
= \;& \mathbf{I}_{p}\left(\int_{r}^{1}tdt-\frac{1}{1-r}\left(\int_{r}^{1}(t-1)^{2}dt+(1-r)^{2}r\right)\right) \\
= \;& \frac{(1-r)^{2}}{6}\mathbf{I}_{p},
\end{aligned}$$

and

$$Var\left[\left(\int_{r}^{1}\widehat{W}_{i}dV_{i}\right)\right]=\frac{(1-r)^{2}}{6}\mathbf{I}_{p}$$

since

$$\begin{aligned}
& E\left[\int_{r}^{1}\left(W_{i}-W_{i}(r)\right)\int_{r}^{1}\left(W_{i}-W_{i}(r)\right)^{'}\right] \\
= \;& \mathbf{I}_{p}\left\{E\left[\int_{r}^{1}W_{i}\int_{r}^{1}W_{i}^{'}-(1-r)W_{i}(r)\int_{r}^{1}W_{i}^{'}-\left(\int_{r}^{1}W_{i}\right)(1-r)W_{i}^{'}(r)+(1-r)^{2}W_{i}(r)W_{i}^{'}(r)\right]\right\} \\
= \;& \mathbf{I}_{p}\left\{\begin{array}{c} E\left[\int_{r}^{1}W_{i}\int_{r}^{1}W_{i}^{'}\right]-(1-r)E\left[W_{i}(r)\int_{r}^{1}W_{i}^{'}\right] \\ -(1-r)E\left[\left(\int_{r}^{1}W_{i}\right)W_{i}^{'}(r)\right]+(1-r)^{2}E\left[W_{i}(r)W_{i}^{'}(r)\right] \end{array}\right\} \\
= \;& \mathbf{I}_{p}\left\{\begin{array}{c} E\left[\int_{r}^{1}W_{i}\int_{r}^{1}W_{i}^{'}\right]-(1-r)\int_{r}^{1}E\left[W_{i}(r)W_{i}^{'}\right] \\ -(1-r)\int_{r}^{1}E\left[W_{i}W_{i}^{'}(r)\right]+(1-r)^{2}E\left[W_{i}(r)W_{i}^{'}(r)\right] \end{array}\right\} \\
= \;& \mathbf{I}_{p}\left\{E\left[\int_{r}^{1}W_{i}\int_{r}^{1}W_{i}^{'}\right]-(1-r)^{2}r\right\} \\
= \;& \mathbf{I}_{p}\int_{r}^{1}(t-1)^{2}dt.
\end{aligned}$$

where

$$\widehat{W}_{i}=W_{i}-\frac{1}{1-r}\int_{r}^{1}W(s).$$

Then we note that

$$\left(\widehat{\beta}_{1k}-\widehat{\beta}_{2k}\right)=\left(\widehat{\beta}_{1k}-\beta\right)-\left(\widehat{\beta}_{2k}-\beta\right)$$

and under H_0

$$\sqrt{n}T\left(\widehat{\beta}_{1k}-\widehat{\beta}_{2k}\right)$$

$$
\begin{aligned}
&= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{n}T\left(\widehat{\beta}_{1k}-\beta\right) \\ \sqrt{n}T\left(\widehat{\beta}_{2k}-\beta\right) \end{bmatrix} \\
&\Rightarrow \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{6}}{r^2}CB(r^2) \\ \frac{\sqrt{6}}{(1-r)^2}C\left[B((1-r)^2)\right] \end{bmatrix} \\
&= \sqrt{6}C\left[\frac{B(r^2)}{r^2}-\frac{B((1-r)^2)}{(1-r)^2}\right] \\
&= \sqrt{6}C\left[\frac{B(1)}{r}-\frac{B(1)}{1-r}\right] \\
&= \sqrt{6}C\left[\frac{B((1-r)^2)-B\left(r^2\right)}{r\left(1-r\right)}\right]
\end{aligned}
$$

uniformly in r. The continuous mapping theorem gives

$$
\begin{aligned}
&\mathrm{W}(k) \\
&= \frac{1}{\widehat{\Omega}_{u.\varepsilon}}\sqrt{n}T\left(\widehat{\beta}_{1k}-\widehat{\beta}_{2k}\right)' \begin{bmatrix} \left(\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{k}(x_{it}-\overline{x}_{1k})^2\right)^{-1} \\ +\left(\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=k+1}^{T}(x_{it}-\overline{x}_{2k})^2\right)^{-1} \end{bmatrix}^{-1} \sqrt{n}T\left(\widehat{\beta}_{1k}-\widehat{\beta}_{2k}\right) \\
&\Rightarrow \frac{1}{\Omega_{u.\varepsilon}}\left[\sqrt{6}C\frac{B((1-r)^2)-B\left(r^2\right)}{r\left(1-r\right)}\right]' \left[\frac{6}{r^2}\Omega_{\varepsilon}^{-1}+\frac{6}{\left(1-r\right)^2}\Omega_{\varepsilon}^{-1}\right]^{-1} \left[\sqrt{6}C\frac{B((1-r)^2)-B\left(r^2\right)}{r\left(1-r\right)}\right] \\
&= \frac{1}{\Omega_{u.\varepsilon}}\left[C\frac{B((1-r)^2)-B\left(r^2\right)}{r\left(1-r\right)}\right]' \left[\frac{1}{r^2}\Omega_{\varepsilon}^{-1}+\frac{1}{\left(1-r\right)^2}\Omega_{\varepsilon}^{-1}\right]^{-1} C\left[\frac{B((1-r)^2)-B\left(r^2\right)}{r\left(1-r\right)}\right] \\
&= \frac{1}{\Omega_{u.\varepsilon}}\left[C\frac{B((1-r)^2)-B\left(r^2\right)}{r\left(1-r\right)}\right]' \frac{r^2\left(1-r\right)^2}{r^2+\left(1-r\right)^2}\Omega_{\varepsilon}C\left[\frac{B((1-r)^2)-B\left(r^2\right)}{r\left(1-r\right)}\right] \\
&= \frac{1}{r^2+\left(1-r\right)^2}\frac{1}{\Omega_{u.\varepsilon}}\left[B((1-r)^2)-B\left(r^2\right)\right]' C'\Omega_{\varepsilon}C\left[B((1-r)^2)-B\left(r^2\right)\right] \\
&= \frac{1}{r^2+\left(1-r\right)^2}\frac{1}{\Omega_{u.\varepsilon}}\left[B((1-r)^2)-B\left(r^2\right)\right]' \Omega_{u.\varepsilon}^{1/2}\Omega_{\varepsilon}^{-1/2}\Omega_{\varepsilon}\Omega_{\varepsilon}^{-1/2}\Omega_{u.\varepsilon}^{1/2}\left[B((1-r)^2)-B\left(r^2\right)\right] \\
&= \frac{1}{r^2+\left(1-r\right)^2}\left[B((1-r)^2)-B\left(r^2\right)\right]'\left[B((1-r)^2)-B\left(r^2\right)\right]
\end{aligned}
$$

uniformly in r, where $Q_p(r^2)$ is the square of a standardized Bessel process of order p. This proves Theorem 1. ■

B Proof of Theorem 2

Proof. The model under the alternative is

$$\begin{aligned} y_{it}^{(T)} &= \alpha + x_{it}^{'}\beta_t^{(T)} + u_{it} \\ &= \alpha + x_{it}^{'}\beta + \frac{1}{T}x_{it}^{'}g\left(\frac{t}{T}\right) + u_{it} \end{aligned}$$

and

$$\left(\widehat{y}_{it}^{+}\right)^T = \left(y_{it}^{*}\right)^T - \widehat{\Omega}_{u\varepsilon}\widehat{\Omega}_{\varepsilon}^{-1}\Delta x_{it}.$$

Note

$$\begin{aligned} \widehat{\beta}_{ik}^{(T)} &= \left[\sum_{i=1}^{n}\sum_{t=1}^{k}(x_{it}-\overline{x}_{1i})(x_{it}-\overline{x}_{1i})^{'}\right]^{-1}\sum_{i=1}^{n}\sum_{t=1}^{k}\left[(x_{it}-\overline{x}_{1i})\left(\widehat{y}_{it}^{+}\right)^T - \widehat{\Delta}_{\varepsilon u}^{+}\right] \\ &= \left[\sum_{i=1}^{n}\sum_{t=1}^{k}(x_{it}-\overline{x}_{1i})(x_{it}-\overline{x}_{1i})^{'}\right]^{-1}\sum_{i=1}^{n}\sum_{t=1}^{k}\left[(x_{it}-\overline{x}_{1i})\left(\left(y_{it}^{*}\right)^T - \widehat{\Omega}_{u\varepsilon}\widehat{\Omega}_{\varepsilon}^{-1}\Delta x_{it}\right) - \widehat{\Delta}_{\varepsilon u}^{+}\right] \\ &= \left[\sum_{i=1}^{n}\sum_{t=1}^{T}(x_{it}-\overline{x}_{1i})(x_{it}-\overline{x}_{1i})^{'}\right]^{-1} \\ &\quad \sum_{i=1}^{n}\sum_{t=1}^{k}\left[(x_{it}-\overline{x}_{1i})\left(\alpha + x_{it}^{'}\beta + \frac{1}{T}x_{it}^{'}g\left(\frac{t}{T}\right) + \widehat{u}_{it}^{+}\right) - \widehat{\Delta}_{\varepsilon u}^{+}\right] \\ &= \left[\sum_{i=1}^{n}\sum_{t=1}^{k}(x_{it}-\overline{x}_{1i})(x_{it}-\overline{x}_{1i})^{'}\right]^{-1} \\ &\quad \sum_{i=1}^{n}\sum_{t=1}^{k}\left[(x_{it}-\overline{x}_{1i})\left(x_{it}^{'}\beta + \frac{1}{T}x_{it}^{'}g\left(\frac{t}{T}\right) + \widehat{u}_{it}^{+}\right) - \widehat{\Delta}_{\varepsilon u}^{+}\right] \\ &= \beta + \left[\sum_{i=1}^{n}\sum_{t=1}^{k}(x_{it}-\overline{x}_{1i})(x_{it}-\overline{x}_{1i})^{'}\right]^{-1}\left[\sum_{i=1}^{n}\sum_{t=1}^{k}(x_{it}-\overline{x}_{1i})\left(\frac{1}{T}x_{it}^{'}g\left(\frac{t}{T}\right) + \widehat{u}_{it}^{+}\right) - \widehat{\Delta}_{\varepsilon u}^{+}\right] \\ &= \beta + \left[\sum_{i=1}^{n}\sum_{t=1}^{k}(x_{it}-\overline{x}_{1i})(x_{it}-\overline{x}_{1i})^{'}\right]^{-1}\left[\sum_{i=1}^{n}\sum_{t=1}^{k}(x_{it}-\overline{x}_{1i})\frac{1}{T}x_{it}^{'}g\left(\frac{t}{T}\right)\right] \\ &\quad + \left[\sum_{i=1}^{n}\sum_{t=1}^{k}(x_{it}-\overline{x}_{1i})(x_{it}-\overline{x}_{1i})^{'}\right]^{-1}\left[\sum_{i=1}^{n}\sum_{t=1}^{k}(x_{it}-\overline{x}_{1i})\widehat{u}_{it}^{+} - \widehat{\Delta}_{\varepsilon u}^{+}\right] \end{aligned}$$

Then

$$\begin{aligned} &\sqrt{n}T\left(\widehat{\beta}_{1k}^{(T)} - \beta\right) \\ =& \left[\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^2}\sum_{t=1}^{k}(x_{it}-\overline{x}_{1i})(x_{it}-\overline{x}_{1i})^{'}\right]^{-1}\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{k}(x_{it}-\overline{x}_{1i})\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)x_{it}\right] \end{aligned}$$

$$+\left[\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^2}\sum_{t=1}^{T}(x_{it}-\overline{x}_{1i})(x_{it}-\overline{x}_{1i})^{'}\right]^{-1}\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{k}(x_{it}-\overline{x}_{1i})\widehat{u}_{it}^{+}-\widehat{\Delta}_{\varepsilon u}^{+}\right]$$

$$= \left[\frac{r^2}{6}\Omega_\varepsilon\right]^{-1}\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{k}(x_{it}-\overline{x}_{1i})\frac{1}{T}x_{it}^{'}g\left(\frac{t}{T}\right)+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{k}(x_{it}-\overline{x}_{1i})\widehat{u}_{it}^{+}-\widehat{\Delta}_{\varepsilon u}^{+}\right]+o_p(1).$$

Similarly

$$\sqrt{n}T\left(\widehat{\beta}_{2k}^{(T)}-\beta\right)$$
$$= \left[\frac{(1-r)^2}{6}\Omega_\varepsilon\right]^{-1}\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=k+1}^{T}(x_{it}-\overline{x}_{2i})\frac{1}{T}x_{it}^{'}g\left(\frac{t}{T}\right)+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=k+1}^{T}(x_{it}-\overline{x}_{2i})\widehat{u}_{it}^{+}-\widehat{\Delta}_{\varepsilon u}^{+}\right]+o_p\left(1\right).$$

This implies that

$$\sqrt{n}T\left(\widehat{\beta}_{1k}^{(T)}-\widehat{\beta}_{2k}^{(T)}\right)$$
$$= \left[\frac{r^2}{6}\Omega_\varepsilon\right]^{-1}\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{k}(x_{it}-\overline{x}_{1i})\frac{1}{T}x_{it}^{'}g\left(\frac{t}{T}\right)+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{k}(x_{it}-\overline{x}_{1i})\widehat{u}_{it}^{+}-\widehat{\Delta}_{\varepsilon u}^{+}\right]$$
$$-\left[\frac{(1-r)^2}{6}\Omega_\varepsilon\right]^{-1}\left[\begin{array}{c}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=k+1}^{T}(x_{it}-\overline{x}_{2i})\frac{1}{T}x_{it}^{'}g\left(\frac{t}{T}\right)\\+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=k+1}^{T}(x_{it}-\overline{x}_{2i})\widehat{u}_{it}^{+}-\widehat{\Delta}_{\varepsilon u}^{+}\end{array}\right]+o_p\left(1\right).$$

From Theorem 1 we know that

$$\begin{aligned}&\left[\frac{r^2}{6}\Omega_\varepsilon\right]^{-1}\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{k}(x_{it}-\overline{x}_{1i})\widehat{u}_{it}^{+}-\widehat{\Delta}_{\varepsilon u}^{+}\right]\\&-\left[\frac{(1-r)^2}{6}\Omega_\varepsilon\right]^{-1}\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=k+1}^{T}(x_{it}-\overline{x}_{2i})\widehat{u}_{it}^{+}-\widehat{\Delta}_{\varepsilon u}^{+}\right]\\ \Rightarrow\quad&\sqrt{6}C\left[\frac{B((1-r)^2)-B\left(r^2\right)}{r\left(1-r\right)}\right]\end{aligned}$$

uniformly in r. It is easy to show that

$$\left[\frac{r^2}{6}\Omega_\varepsilon\right]^{-1}\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{k}(x_{it}-\overline{x}_{1i})x_{it}^{'}\frac{1}{T}g\left(\frac{t}{T}\right)\right]$$
$$-\left[\frac{(1-r)^2}{6}\Omega_\varepsilon\right]^{-1}\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=k+1}^{T}(x_{it}-\overline{x}_{2i})x_{it}^{'}\frac{1}{T}g\left(\frac{t}{T}\right)\right]=O_p\left(1\right)$$

uniformly in r using (e.g., Bai, 1996, p. 609) for a fixed n,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{k}(x_{it}-\overline{x}_{1i})x_{it}^{'}\frac{1}{T}g\left(\frac{t}{T}\right)$$

$$\begin{aligned}
&\Rightarrow \quad \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \int_{0}^{r} \frac{\Omega_{\varepsilon}^{1/2} d\left(\int_{0}^{s} \widetilde{W}_i \widetilde{W}_i'\right) \Omega_{\varepsilon}^{1/2}}{ds} g(s)\, ds \\
&= \quad \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \int_{0}^{r} \Omega_{\varepsilon}^{1/2} \widetilde{W}_i \widetilde{W}_i' \Omega_{\varepsilon}^{1/2} g(s)\, ds,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{T} \sum_{t=k+1}^{T} (x_{it} - \overline{x}_{2i}) x_{it}' \frac{1}{T} g\left(\frac{t}{T}\right) \\
&\Rightarrow \quad \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \int_{r}^{1} \frac{\Omega_{\varepsilon}^{1/2} d\left(\int_{s}^{1} \widehat{W}_i \widehat{W}_i{}'\right) \Omega_{\varepsilon}^{1/2}}{ds} g(s)\, ds \\
&= \quad -\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \int_{r}^{1} \Omega_{\varepsilon i}^{1/2} \widehat{W}_i \widehat{W}_i \Omega_{\varepsilon}^{1/2} g(s)\, ds,
\end{aligned}$$

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \int_{0}^{r} \Omega_{\varepsilon}^{1/2} \widetilde{W}_i \widetilde{W}_i' \Omega_{\varepsilon}^{1/2} g(s)\, ds = O_p(1)$$

and

$$-\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \int_{r}^{1} \Omega_{\varepsilon}^{1/2} \widetilde{W}_i \widetilde{W}_i' \Omega_{\varepsilon}^{1/2} g(r)\, dr = O_p(1)$$

uniformly in r. It follows that

$$\begin{aligned}
&\sqrt{n} T \left(\widehat{\beta}_{1k}^{(T)} - \widehat{\beta}_{2k}^{(T)}\right) \\
&\Rightarrow \quad \sqrt{6} C \frac{B((1-r)^2) - B\left(r^2\right)}{r(1-r)} + O_p(1). \qquad (24)
\end{aligned}$$

Then under the alternative

$$\begin{aligned}
& \mathrm{W}(k) \\
= \quad & \frac{1}{\widetilde{\Omega}_{u.\varepsilon}} \sqrt{n} T \left(\widehat{\beta}_{1k}^{(T)} - \widehat{\beta}_{2k}^{(T)}\right)' \begin{bmatrix} \left(\frac{1}{n}\sum_{i=1}^{n} \frac{1}{T^2} \sum_{t=1}^{k} (x_{it} - \overline{x}_{1ik})^2\right)^{-1} \\ + \left(\frac{1}{n}\sum_{i=1}^{n} \frac{1}{T^2} \sum_{t=k+1}^{T} (x_{t} - \overline{x}_{2ik})^2\right)^{-1} \end{bmatrix}^{-1} \sqrt{n} T \left(\widehat{\beta}_{1k}^{(T)} - \widehat{\beta}_{2k}^{(T)}\right) \\
\Rightarrow \quad & \frac{1}{\Omega_{u.\varepsilon}} \left[\sqrt{6} C \left[\frac{B((1-r)^2) - B\left(r^2\right)}{r(1-r)}\right] + O_p(1)\right]' \left[6 \Omega_{\varepsilon}^{-1} \frac{(1-r)^2 + r^2}{r^2 (1-r)^2}\right]^{-1} \left[\sqrt{6} C \left[\frac{B((1-r)^2) - B\left(r^2\right)}{r(1-r)}\right] + O_p(1)\right] \\
= \quad & \frac{\Omega_{u.\varepsilon}}{\Omega_{u.\varepsilon}} Q_p(r) + O_p(1) \\
= \quad & Q_p(r) + O_p(1)
\end{aligned}$$

uniformly in r using (24) and

$$\widetilde{\Omega}_{u.\varepsilon} = \widehat{\Omega}_{u.\varepsilon} + o_p(1).$$

This proves Theorem 2. ■

C Proof of Theorem 3

Proof. We note from Theorem 1 and Kao (1999) that

$$\sqrt{n}\left(\widehat{\beta}_{1k}-\beta\right) \Rightarrow \frac{\sqrt{\frac{2}{5}}}{r^2} C B(r^2), \tag{25}$$

$$\sqrt{n}\left(\widehat{\beta}_{2k}-\beta\right) \Rightarrow \frac{\sqrt{\frac{2}{5}}}{(1-r)^2} C \left(B((1-r)^2)\right), \tag{26}$$

and

$$\sqrt{n}\left(\widehat{\beta}_{1k}-\widehat{\beta}_{2k}\right) \Rightarrow \sqrt{\frac{2}{5}} C \left[\frac{B((1-r)^2)-B\left(r^2\right)}{r\left(1-r\right)}\right], \tag{27}$$

where $C=\left(\Omega_{\varepsilon}^{-1}\Omega_{u.\varepsilon}\right)^{1/2}$ and $B(r^2)$ is a $p\times 1$ vector of independent Brownian motion. Combining (25) - (27) and the continuous mapping theorem we have

$$\begin{aligned}
& \frac{1}{T^2}\mathrm{W}(k) \\
= \quad & \frac{1}{\widehat{\Omega}_{u.\varepsilon}}\sqrt{n}\left(\widehat{\beta}_{1k}-\widehat{\beta}_{2k}\right)' \left[\begin{array}{c} \left(\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{k}(x_{it}-\overline{x}_{1k})^2\right)^{-1} \\ +\left(\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=k+1}^{T}(x_{it}-\overline{x}_{2k})^2\right)^{-1}\end{array}\right]^{-1} \sqrt{n}\left(\widehat{\beta}_{1k}-\widehat{\beta}_{2k}\right) \\
\Rightarrow \quad & \frac{1}{\Omega_{u.\varepsilon}}\left[\sqrt{\frac{2}{5}}C\left[\frac{B((1-r)^2)-B\left(r^2\right)}{r\left(1-r\right)}\right]\right]' \left[6\Omega_{\varepsilon}^{-1}\frac{(1-r)^2+r^2}{r^2\left(1-r\right)^2}\right]^{-1} \left[\sqrt{\frac{2}{5}}C\left[\frac{B((1-r)^2)-B\left(r^2\right)}{r\left(1-r\right)}\right]\right] \\
= \quad & \frac{1}{15\Omega_{u.\varepsilon}}\frac{r^2\left(1-r\right)^2}{(1-r)^2+r^2}\left[\frac{B((1-r)^2)-B\left(r^2\right)}{r\left(1-r\right)}\right]' C'\Omega_{\varepsilon}C\left[\frac{B((1-r)^2)-B\left(r^2\right)}{r\left(1-r\right)}\right] \\
= \quad & \frac{1}{15}\frac{1}{\Omega_{u.\varepsilon}}\frac{\left[B((1-r)^2)-B\left(r^2\right)\right]'\Omega_{u.\varepsilon}^{1/2}\Omega_{\varepsilon}^{-1/2}\Omega_{\varepsilon}\Omega_{\varepsilon}^{-1/2}\Omega_{u.\varepsilon}^{1/2}\left[B((1-r)^2)-B\left(r^2\right)\right]}{(1-r)^2+r^2} \\
= \quad & \frac{1}{15}\frac{\Omega_{u.\varepsilon}}{\Omega_{u.\varepsilon}}\frac{\left[B((1-r)^2)-B\left(r^2\right)\right]'\left[B((1-r)^2)-B\left(r^2\right)\right]}{(1-r)^2+r^2} \\
= \quad & \frac{1}{15}\frac{\Omega_{u.\varepsilon}}{\Omega_{u.\varepsilon}}\frac{\left[B((1-r)^2)-B\left(r^2\right)\right]'\left[B((1-r)^2)-B\left(r^2\right)\right]}{(1-r)^2+r^2} \\
= \quad & \frac{1}{15}Q_p(r)
\end{aligned}$$

uniformly in r which completes the proof of Theorem 3. ■

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