Testing for Structural Change of a Time Trend Regression in Panel Data

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Abstract

In this paper, we propose two classes of test statistics for detecting a break at an unknown date in panel data models with time trend. The first one is the fluctuation test of Ploberger-Kramer-Kontrus (1989). The second one is based on the mean and exponential Wald statistics of Andrew and Ploberger (1994) and maximum Wald statistic of Andrew (1993). We derive the limiting distributions of the proposed tests and tabulate the critical values. Asymptotic results were derived $I(0)$, $I(1)$ and nearly $I(1)$ error terms. Monte Carlo simulations are performed to examine the size and power of the proposed tests

1 Introduction

Testing for structural changes has been an important research topic in nonstationary time series econometrics. Recent issues of the *Journal of Business and Economic Statistics* and *Journal of Econometrics* are devoted to such studies, e.g., Chu and White (1992); Hansen (1992); Gregory and Hansen (1996); Campos, Ericsson and Hendry (1996). Kao and Ross (1995) extended the dynamic cumulative sum (CUSUM) test of Kramer, Ploberger and Alt (1988) to the model where serial correlation is present. None of these papers has looked at the tests in the context of the panel data except Han and Park (1989) and Hansen (1999). Han and Park (1989) proposed a CUSUM and a CUSUM of squares tests for panel data models. Hansen (1999) developed methods for testing the threshold effects in panel data. In a recent paper, though not a panel context, Bai, Lumsdaine and Stock (1998) developed methods for testing and constructing asymptotically valid confidence intervals for the date of a single break in multivariate time series, including $I(0)$, $I(1)$ and deterministically trending regressors. They showed there are substantial gains by using multivariate time series which have a common break date.

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This paper, along with Kao and Chiang (2000), is a first step in understanding how to test for structural changes when nonstationary panel data is being used. In this paper, we propose two classes of test statistics for detecting a break at an unknown date in panel data models with time trend. The first one is the fluctuation test of Ploberger-Kramer-Kontrus (1989). The second one is based on the mean and exponential Wald statistics of Andrew and Ploberger (1994) and maximum Wald statistic of Andrew (1993). We derive the limiting distributions of the proposed tests and tabulate the critical values. Asymptotic results were derived for $I(0)$, $I(1)$ and nearly $I(1)$ error terms. We also show that these tests have non-trivial local power.

This paper contributes to the literature of testing for structural changes in two ways. First, we extend the fluctuations tests of Chu and White (1992) and Wald tests of Vogelsang (1997) to panel models. Second, it provides a serious study of the finite sample properties of the proposed tests.

The organization of the paper is as follows. Section 2 introduces the model and test statistics. The limiting distributions of the proposed test statistics with an $I(0)$ error term under the null hypothesis are established. Section 3 gives the limiting distributions of the test statistics under the null hypothesis with an $I(1)$ error term. In Section 4, we discuss the limiting distributions of the test statistics under the null hypothesis when the error is nearly $I(1)$. Section 5 establishes the limiting distributions of test statistics under local alternatives. In section 6 we derive the limiting distributions of the test statistics under both the null hypothesis and the local alternatives for a polynomial trend model. Section 7 presents Monte Carlo results to evaluate the finite sample properties of the proposed test statistics. In Section 8 we summarize the findings. All proofs are in the Appendix.

A word on notation. We use $\xrightarrow{d}$ to denote convergence in distribution, $\xrightarrow{p}$ to denote convergence in probability, $[x]$ to denote the largest integer $\leq x$, and $I(0)$ and $I(1)$ to signify a time series that is integrated of order zero and one, respectively.

2 The Model and the Tests

Consider the following simple linear trend with one-way error component model

$$y_{it} = \alpha + \beta_t t + u_{it}, \tag{1}$$

$$u_{it} = \mu_i + v_{it},$$

$i = 1, ..., N$, $t = 1, ...T$, where $\{y_{it}\}$ are 1×1, β is the slope parameters, $\{\mu_i\}$ are the unobservable individual effects with $\mu_i \sim iid\left(0, \sigma_\mu^2\right)$, and $\{v_{it}\}$ are AR(1) stationary disturbance terms with

$$v_{it} = \rho v_{it-1} + \varepsilon_{it}, |\rho| < 1, \tag{2}$$

where $\varepsilon_{it} \sim iid\left(0, \sigma_\varepsilon^2\right)$. The μ_i are assumed to be independent of v_{it} and $v_{it} \sim \left(0, \sigma_v^2\right)$, $t = 2, ..., T$, where $\sigma_v^2 = \frac{\sigma_\varepsilon^2}{1-\rho^2}$. We assume $v_{i1} = \sum_{j=0}^{[\kappa T]} \rho^j \varepsilon_{i1-j}$, where κ is a parameter that governs the variance of the initial condition. When $\kappa = 0$, v_{i1} is $O_p(1)$. When $\kappa > 0$, v_{i1} is $O_p(1)$ when v_{it} is $I(0)$ but is $O_p(T^{1/2})$ when v_{it} is $I(1)$.

The problem of interest is to test the changes in the parameter β where the change points are unknown. For testing the null hypothesis

$$H_0 : \beta_t = \beta \text{ for all } t. \tag{3}$$

The estimator to be considered is the recursive OLS

$$\widehat{\beta}_k = \frac{\sum_{i=1}^N \left[\sum_{t=1}^k (t - \bar{t}_k) y_{it}\right]}{\sum_{i=1}^N \sum_{t=1}^k (t - \bar{t}_k)^2}, \tag{4}$$

where

$$\bar{t}_k = \frac{1}{k} \sum_{t=1}^k t.$$

Following Ploberger et al. (1989) and Chu and White (1992), the null hypothesis is rejected if $\widehat{\beta}_k$ fluctuate too much, i.e., the null hypothesis is rejected if

$$\max_{i=1,\ldots,k} \left|\widehat{\beta}_k - \widehat{\beta}_T\right|$$

is too large. Define the test statistic to be

$$T_1 = \sup_{2 \le k \le T-1} \left| \sqrt{NT^3} \frac{1}{6\sigma_0} \left(\frac{k}{T}\right)^3 \left(\widehat{\beta}_k - \widehat{\beta}_T\right) \right|, \tag{5}$$

where

$$\sigma_0^2 = \frac{\sigma_\varepsilon^2}{(1-\rho)^2}. \tag{6}$$

All limits in Theorems 1-3, 5, 7-8, 10, and Lemma 1 are taken as $T \to \infty$ for a fixed N except Theorems 4, 6, 9 which are taken as $T \to \infty$ followed by $N \to \infty$ sequentially. Also all the convergences in all the theorems are uniform convergence in r. We then prove the following theorem:

Theorem 1 *Under H_0.*

$$\sqrt{NT^3}\frac{1}{6\sigma_0}\left(\frac{k}{T}\right)^3\left(\widehat{\beta}_k-\widehat{\beta}_T\right)\xrightarrow{d}G_0(r),$$

where

$$G_0(r)=G(r)-r^3G(1).$$

and

$$G(r)=rW(r)-2\int_0^r W(s)ds.$$

Theorem 1 provides the limiting distribution of the test statistic in (5). Since

$$\begin{aligned} P\left(T_1>c\right) &\rightarrow P\left(\sup_{r^*\leq r\leq 1-r^*}|G_0(r)|>c\right) \\ &= P\left(\sup_{s^*\leq s\leq 1-s^*}|W_0(s)|>\sqrt{3}c\right) \end{aligned} \tag{7}$$

under H_0, where $W_0(s)$ is a standard Brownian bridge. Note $P\left(\sup_{s^*\leq s\leq 1-s^*}|W_0(s)|>\sqrt{3}c\right)$ is well known (e.g., Chu and White, 1992). Some useful critical values for are .708 (10%), .784 (5%), and .940 (1%).

Consider the alternative hypothesis that there is only one change point k, i.e.,

$$H_1:\beta_t=\begin{cases}\beta_1 & \text{for } t=1,...,k\\ \beta_2 & \text{for } t=k+1,...,T\end{cases} \tag{8}$$

Let $W(k)$ be the Wald statistic for testing $\beta_1=\beta_2$:

$$\begin{aligned} \mathrm{W}(k) &= \frac{1}{\sigma_v^2}\left(\widehat{\beta}_{1k}-\widehat{\beta}_{2k}\right)'\left[\left(\sum_{i=1}^N\sum_{t=1}^k(t-\bar{t}_{1k})^2\right)^{-1}+\left(\sum_{i=1}^N\sum_{t=k+1}^T(t-\bar{t}_{2k})^2\right)^{-1}\right]^{-1}\left(\widehat{\beta}_{1k}-\widehat{\beta}_{2k}\right) \\ &= \frac{1}{\sigma_v^2}\frac{\left(\widehat{\beta}_{1k}-\widehat{\beta}_{2k}\right)^2}{\left[\left(\sum_{i=1}^N\sum_{t=1}^k(t-\bar{t}_{1k})^2\right)^{-1}+\left(\sum_{i=1}^N\sum_{t=k+1}^T(t-\bar{t}_{2k})^2\right)^{-1}\right]}, \end{aligned} \tag{9}$$

where

$$\widehat{\beta}_{1k}=\frac{\sum_{i=1}^N\left[\sum_{t=1}^k(t-\bar{t}_{1k})y_{it}\right]}{\sum_{i=1}^N\sum_{t=1}^k(t-\bar{t}_{1k})^2},$$

$$\widehat{\beta}_{2k}=\frac{\sum_{i=1}^N\left[\sum_{t=k+1}^T(t-\bar{t}_{2k})y_{it}\right]}{\sum_{i=1}^N\sum_{t=k+1}^T(t-\bar{t}_{2k})^2},$$

$$\bar{t}_{1k} = \frac{1}{k}\sum_{t=1}^{k} t,$$

and

$$\bar{t}_{2k} = \frac{1}{T-k}\sum_{t=k+1}^{T} t.$$

Define the following statistic:

$$\mathrm{W}_1(k) = \frac{\sigma_v^2}{3\sigma_0^2}\mathrm{W}(k).$$

Then we have the following theorem:

Theorem 2 *Under H_0 :*

$$\mathrm{W}_1(k) \xrightarrow{d} Q_1(r)$$

where

$$Q_1(r) = \left\{ \frac{G(r)(1-r)^3 - r^3\left[G(1) - G(r) + W(r) - rW(1)\right]}{\left[r^3(1-r)^3\left[(1-r)^3 + r^3\right]\right]^{\frac{1}{2}}} \right\}^2 .$$

Following Vogelsang (1997), we consider three statistics: $sup\mathrm{W}_1(k)$, $Mean\mathrm{W}_1(k)$, and $Exp\mathrm{W}_1(k)$, where

$$sup\mathrm{W}_1(k) = \sup_{[Tr^*]\leq k\leq T-[Tr^*]} \mathrm{W}_1(k),$$

$$Mean\mathrm{W}_1(k) = \frac{1}{T}\sum_{k=[Tr^*]}^{T-[Tr^*]} \mathrm{W}_1(k),$$

$$Exp\mathrm{W}_1(k) = \log\left(\frac{1}{T}\sum_{k=[Tr^*]}^{T-[Tr^*]} \exp\left(\frac{1}{2}\mathrm{W}_1(k)\right)\right),$$

and r^* is the fraction of trimming, usually taken to be either 0.15 or 0.01. Using the continuous mapping theorem we then have the following corollary:

Corollary 1 *Under H_0 :*

1. $sup\mathrm{W}_1(k) \xrightarrow{d} \sup_{r^*\leq r\leq 1-r^*} Q_1(r),$

2. $Mean\mathrm{W}_1(k) \xrightarrow{d} \int_{r^*}^{1-r^*} Q_1(r)dr$,

3. $Exp\mathrm{W}_1(k) \xrightarrow{d} \log\left(\int_{r^*}^{1-r^*} \exp\left(\frac{1}{2}Q_1(r)\right) dr\right)$.

Remark 1 *1. In practice, we need to replace σ_ε^2 by a consistent estimator, $\widehat{\sigma}_\varepsilon^2$ (e.g., Baltagi, 1991). The limiting distributions of T_1 and $\mathrm{W}_1(k)$ will be unchanged if we replace σ_ε^2 by $\widehat{\sigma}_\varepsilon^2$.*

2. The results in this section will not change if we replace the AR(1) assumption in (2) by a more general process, e.g.,

$$v_{it} = \psi(L)\varepsilon_{it} = \sum_{j=0}^{\infty} \psi_j \varepsilon_{it-j},$$

where $\sum_{j=0}^{\infty} j\left|\psi_j\right| < \infty$ and $\varepsilon_{it} \sim iid\left(0, \sigma_\varepsilon^2\right)$. Then $\sigma_0^2 = \sigma_\varepsilon^2 \psi^2(1)$. Note $\psi(1) = \frac{1}{1-\rho}$ if v_{it} is assumed to be an AR(1) in (2).

3. Also the results of this section will not change if we replace iid by martingale difference sequence (MDS) for ε_{it}.

4. The homogeneity assumption made for σ_ε^2 and ρ across i can be relaxed by allowing $\sigma_{\varepsilon i}^2$ and ρ_i to differed for different i. The limiting distributions of T_1 and $\mathrm{W}_1(k)$ will be unchanged if we define σ_0^2 and σ_v^2 as follows

$$\sigma_0^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \frac{\sigma_{\varepsilon i}^2}{\left(1-\rho_i\right)^2}$$

and

$$\sigma_v^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \frac{\sigma_{\varepsilon i}^2}{1-\rho_i^2}.$$

3 The Limiting Distribution of the Test Statistics when $\rho = 1$

Model (2) is restrictive because it excludes v_{it} to be $I(1)$. We investigated the asymptotic properties of the two test statistics, T_1 and $\mathrm{W}_1(k)$, in Section 2. In this section v_{it} is $I(1)$. We will show that the previous conclusions in Section 2 are substantially altered when v_{it} is $I(1)$. Define

$$T_2 = \sup_{2\le k\le T-1} \left|\sqrt{NT}\frac{1}{6\sigma_\varepsilon}\left(\frac{k}{T}\right)^3\left(\widehat{\beta}_k - \widehat{\beta}_T\right)\right|. \qquad (10)$$

Theorem 3 *Under H_0 and $v_{it} = v_{it-1} + \varepsilon_{it}$ then*

$$\sqrt{NT}\frac{1}{6\sigma_\varepsilon}\left(\frac{k}{T}\right)^3\left(\widehat{\beta}_k - \widehat{\beta}_T\right) \xrightarrow{d} H_0(r),$$

where

$$H_0(r) = H(r) - r^3 H(1),$$

$$H(r) = 2\int_0^r s\left[W(s) + \widetilde{W}(\kappa)\right]ds - r\int_0^r \left[W(s) + \widetilde{W}(\kappa)\right]ds,$$

and

$$H(1) = 2\int_0^1 s\left[W(s) + \widetilde{W}(\kappa)\right]ds - \int_0^1 \left[W(s) + \widetilde{W}(\kappa)\right]ds.$$

From Theorem 2 we know that the limiting distribution of T_2 differs from T_1.

Next, define

$$\mathrm{W}_2(k) = \frac{1}{T^2}\frac{\sigma_v^2}{3\sigma_\varepsilon^2}\mathrm{W}(k).$$

Theorem 4 *Under H_0 and $v_{it} = v_{it-1} + \varepsilon_{it}$ then*

$$\mathrm{W}_2(k) \xrightarrow{d} Q_2(r)$$

where

$$Q_2(r) = \left\{\frac{H(r)(1-r)^3 - \left[H(1) - H(r) - r\int_0^1 W(s)ds + \int_0^r W(s)ds\right]r^3}{\left[r^3(1-r)^3\left[(1-r)^3 + r^3\right]\right]^{1/2}}\right\}^2$$

Using the continuous mapping theorem we then have following corollary:

Corollary 2 *Under H_0 :*

1. $sup\mathrm{W}_2(k) \xrightarrow{d} \underset{r^* \leq r \leq 1-r^*}{sup} Q_2(r)$,
2. $Mean\mathrm{W}_2(k) \xrightarrow{d} \int_{r^*}^{1-r^*} Q_2(r)dr$,
3. $Exp\mathrm{W}_2(k) \xrightarrow{d} \log\left(\int_{r^*}^{1-r^*} \exp\left(\frac{1}{2}Q_2(r)\right)dr\right)$.

Remark 2 *In practice, we need to replace σ_ε^2 and κ with consistent estimators, $\widehat{\sigma}_\varepsilon^2$ and $\widehat{\kappa}$. The limiting distributions of T_2 and $\mathrm{W}_2(k)$ will be unchanged if we replace σ_ε^2 and κ by $\widehat{\sigma}_\varepsilon^2$ and $\widehat{\kappa}$. $\widehat{\sigma}_\varepsilon^2$ can be formed by*

$$\widehat{\sigma}_\varepsilon^2 = \frac{1}{N(T-1)}\sum_{i=1}^N\sum_{t=2}^T\left(u_{it} - u_{it-1}\right)^2.$$

Moon and Phillips (1999) proposed a consistent estimator for κ.

4 Nearly I(1) Errors

In recent years, there has been considerable interest in the asymptotic properties of the estimation and inference of β in (1) when ρ is close to one in the time-series (i.e., when $N = 1$) econometrics literature. In this section we assume $\rho = 1 + c/T$ in (2), i.e., the v_{it} follows a local-to-unit or a nearly $I(1)$ process. The asymptotics for the test statistic T_1 are given in the following theorem:

Theorem 5 *Under H_0 and $v_{it} = \rho v_{it-1} + \varepsilon_{it}$, $\rho = 1 + c/T$, then*

$$\sqrt{NT}\frac{1}{6\sigma_\varepsilon}\left(\frac{k}{T}\right)^3\left(\widehat{\beta}_k - \widehat{\beta}_T\right) \xrightarrow{d} H_c(r) - r^3 H_c(1),$$

where

$$H_c(r) = 2\int_0^r s\left[W_c(s) + e^{sc}\tilde{W}_c(\kappa)\right] ds - r\int_0^r \left[W_c(s) + e^{sc}\tilde{W}_c(\kappa)\right] ds$$

and

$$H_c(1) = 2\int_0^1 s\left[W_c(s) + e^{sc}\tilde{W}_c(\kappa)\right] ds - \int_0^1 \left[W_c(s) + e^{sc}\tilde{W}_c(\kappa)\right] ds.$$

Next we present the limiting distribution of the Wald Statistic when $\rho = 1+c/T$. Define the test statistic

$$W_c(k) = \frac{1}{T^2}\frac{\sigma_v^2}{3\sigma_\varepsilon^2}\mathrm{W}(k).$$

Theorem 6 *Under H_0 and $v_{it} = \rho v_{it-1} + \varepsilon_{it}$, $\rho = 1 + c/T$, then*

$$\frac{1}{T^2}W_c(k) \xrightarrow{d} Q_c(r),$$

where

$$Q_c(r) = \left\{\frac{H_c(r)(1-r)^3 - \left[H_c(1) - H_c(r) - r\int_0^1 \left[W_c(s) + e^{sc}\tilde{W}_c(\kappa)\right] ds + \int_0^r \left[W_c(s) + e^{sc}\tilde{W}_c(\kappa)\right] ds\right] r^3}{\left[r^3(1-r)^3\left[(1-r)^3 + r^3\right]\right]^{1/2}}\right\}^2.$$

Remark 3 *Note*

$$\lim_{c\to 0}\left(H_c(r) - r^3 H_c(1)\right) = H(r) - r^3 H(1)$$

and

$$\lim_{c\to 0} Q_c(r) = Q_2(r).$$

5 Local Asymptotic Power

Consider the local alternative:

$$\beta_t^{(T)} = \beta + \frac{1}{\sqrt{T}} g\left(\frac{t}{T}\right), \tag{11}$$

where g is an arbitrary function which has bounded variation on $[0,1]$. Define

$$y_{it}^{(T)} = \alpha + \beta_t^{(T)} t + u_{it}$$

and let

$$\widehat{\beta}_k^{(T)} = \frac{\sum_{i=1}^N \sum_{t=1}^k (t - \bar{t}_{1k}) y_{it}^{(T)}}{\sum_{i=1}^N \sum_{t=1}^k (t - \bar{t}_{1k})^2}$$

be the OLS estimator under the local alternative (11). Similarly, let

$$T_1^{(T)} = \sup_{2 \le k \le T-1} \left| \sqrt{NT^3} \frac{1}{6\sigma_0} \left(\frac{k}{T}\right)^3 \left(\widehat{\beta}_k^{(T)} - \widehat{\beta}_T^{(T)}\right) \right|,$$

$$T_2^{(T)} = \sup_{2 \le k \le T-1} \left| \sqrt{NT} \frac{1}{6\sigma_\varepsilon} \left(\frac{k}{T}\right)^3 \left(\widehat{\beta}_k^{(T)} - \widehat{\beta}_T^{(T)}\right) \right|,$$

$$\mathrm{W}^{(T)}(k) = \frac{1}{\sigma_v^2} \frac{\left(\widehat{\beta}_{1k}^{(T)} - \widehat{\beta}_{2k}^{(T)}\right)^2}{\left[\left(\sum_{i=1}^N \sum_{t=1}^k (t - \bar{t}_{1k})^2\right)^{-1} + \left(\sum_{i=1}^N \sum_{t=k+1}^T (t - \bar{t}_{2k})^2\right)^{-1}\right]},$$

$$\mathrm{W}_1^{(T)}(k) = \frac{\sigma_v^2}{3\sigma_0^2} \mathrm{W}^{(T)}(k),$$

$$\mathrm{W}_c^{(T)}(k) = \frac{1}{T^2} \frac{\sigma_v^2}{3\sigma_\varepsilon^2} \mathrm{W}^{(T)}(k)$$

and

$$\mathrm{W}_2^{(T)}(k) = \frac{1}{T^2} \frac{\sigma_v^2}{3\sigma_\varepsilon^2} \mathrm{W}^{(T)}(k)$$

be the corresponding test statistics under local alternative.

Theorem 7 *Under the local alternatives (11),*

1. *If* $|\rho| < 1$, *then*

$$\sqrt{NT^3}\frac{1}{6\sigma_0}\left(\frac{k}{T}\right)^3\left(\widehat{\beta}_k^{(T)} - \widehat{\beta}_T^{(T)}\right) \xrightarrow{d} G_0(r) + O_p\left(T\right),$$

2. *If* $\rho = 1$, *then*

$$\sqrt{NT}\frac{1}{6\sigma_\varepsilon}\left(\frac{k}{T}\right)^3\left(\widehat{\beta}_k^{(T)} - \widehat{\beta}_T^{(T)}\right) \xrightarrow{d} H_0(r) + \frac{2}{\sigma_\varepsilon}h_0\left(r\right),$$

3. *If* $\rho = 1 + \frac{c}{T}$, *then*

$$\sqrt{NT}\frac{1}{6\sigma_\varepsilon}\left(\frac{k}{T}\right)^3\left(\widehat{\beta}_k^{(T)} - \widehat{\beta}_T^{(T)}\right) \xrightarrow{d} H_c(r) - r^3H_c(1) + \frac{2}{\sigma_\varepsilon}h_0\left(r\right),$$

where

$$h_0\left(r\right) = h(r) - r^3h(1),$$

$$h(r) = \int_0^r s^2g(s)ds - \frac{1}{2}r\int_0^r sg(s)ds,$$

and

$$h(1) = \int_0^1 s^2g(s)ds - \frac{1}{2}\int_0^1 sg(s)ds.$$

If $h_0\left(r\right) = 0$ we obtain the distribution under the null. Next we consider the behavior of $W(k)$ under sequences of local alternatives.

Theorem 8 *Under the local alternatives (11),*

1. *If* $|\rho| < 1$, *then*

$$W_1^{(T)}(k) \xrightarrow{d} \left\{\left[Q_1(r)\right]^{\frac{1}{2}} + O_p\left(T\right)\right\}^2,$$

2. *If* $\rho = 1$, *then*

$$W_2^{(T)}(k) \xrightarrow{d} \left\{\left[Q_2(r)\right]^{\frac{1}{2}} + \frac{2}{\sigma_\varepsilon}\left[Q_3(r)\right]^{\frac{1}{2}}\right\}^2,$$

3. *If* $\rho = 1 + \frac{c}{T}$, *then*

$$W_c^{(T)}(k) \xrightarrow{d} \left\{\left[Q_c(r)\right]^{\frac{1}{2}} + \frac{2}{\sigma_\varepsilon}\left[Q_3(r)\right]^{\frac{1}{2}}\right\}^2,$$

where

$$Q_3(r) = \left\{\frac{(1-r)^3h(r) - r^3\left[h(1) - h(r) - \frac{1}{2}r\int_0^1 sg(s)ds + \frac{1}{2}\int_0^r sg(s)ds\right]}{\left[r^3(1-r)^3\left[(1-r)^3 + r^3\right]\right]^{\frac{1}{2}}}\right\}^2.$$

6 Polynomial Trend

Consider the panel polynomial regression

$$y_{it} = \alpha + \beta_{1t} t + \cdots + \beta_{pt} t^p + u_{it}. \tag{12}$$

The null hypothesis is $\beta_{1t} = \beta_1$, $\beta_{2t} = \beta_2, \ldots$, and $\beta_{pt} = \beta_p$. Let $W_{pT}(k)$ be the Wald statistic for testing the null hypothesis.

$$\begin{aligned} W_{pT}(k) &= (Rb-q)' \left[var(Rb-q)\right]^{-1} (Rb-q) \\ &= \left(\widehat{\underline{\beta}}_{(1k)} - \widehat{\underline{\beta}}_{(2k)}\right)' \left[\sigma_v^2 R(Var(b_a))R'\right]^{-1} \left(\widehat{\underline{\beta}}_{(1k)} - \widehat{\underline{\beta}}_{(2k)}\right) \\ &= \frac{1}{\sigma_v^2} \left(\widehat{\underline{\beta}}_{(1k)} - \widehat{\underline{\beta}}_{(2k)}\right)' \left[R(Var(b_a))R'\ \right]^{-1} \left(\widehat{\underline{\beta}}_{(1k)} - \widehat{\underline{\beta}}_{(2k)}\right), \end{aligned}$$

where

$$\begin{aligned} b &= \left(\widehat{\beta}_{1(1k)}, \widehat{\beta}_{2(1k)}, \ldots, \widehat{\beta}_{p(1k)}, \widehat{\beta}_{1(2k)}, \widehat{\beta}_{2(2k)}, \ldots, \widehat{\beta}_{p(2k)}\right)' \\ &= \left(\widehat{\underline{\beta}}_{(1k)}, \widehat{\underline{\beta}}_{(2k)}\right)' \end{aligned}$$

is a $2p \times 1$ vector,

$$Var\,(b_a) = \begin{bmatrix} \left[\sum_{i=1}^N \sum_{t=1}^k (\underline{t} - \overline{\underline{t}}_{(1k)})'(\underline{t} - \overline{\underline{t}}_{(1k)})\right]^{-1} & 0_{p\times p} \\ 0_{p\times p} & \left[\sum_{i=1}^N \sum_{t=k+1}^T (\underline{t} - \overline{\underline{t}}_{(2k)})'(\underline{t} - \overline{\underline{t}}_{(2k)})\right]^{-1} \end{bmatrix}$$

is a $2p \times 2p$ matrix, and

$$R = \begin{bmatrix} 1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \vdots & 0 & -1 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 & \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -1 \end{bmatrix}$$

is a $p \times 2p$ matrix. Also,

$$\widehat{\underline{\beta}}_{(1k)} = \left[\sum_{i=1}^N \sum_{t=1}^k (\underline{t} - \overline{\underline{t}}_{(1k)})'(\underline{t} - \overline{\underline{t}}_{(1k)})\right]^{-1} \left[\sum_{i=1}^N \sum_{t=1}^k (\underline{t} - \overline{\underline{t}}_{(1k)})' y_{it}\right],$$

$$\widehat{\underline{\beta}}_{(2k)} = \left[\sum_{i=1}^N \sum_{t=k+1}^T (\underline{t} - \overline{\underline{t}}_{(2k)})'(\underline{t} - \overline{\underline{t}}_{(2k)})\right]^{-1} \left[\sum_{i=1}^N \sum_{t=k+1}^T (\underline{t} - \overline{\underline{t}}_{(2k)})' y_{it}\right],$$

where

$$\underline{t} = (t, \dots, t^p) \ ,$$

$$\overline{\underline{t}}_{(1k)} = \left(\frac{1}{k} \sum_{t=1}^{k} t, \cdots, \frac{1}{k} \sum_{t=1}^{k} t^p \right) \ ,$$

and

$$\overline{\underline{t}}_{(2k)} = \left(\frac{1}{T-k} \sum_{t=k+1}^{T} t, \dots, \frac{1}{T-k} \sum_{t=k+1}^{T} t^p \right).$$

More specifically, we calculate the Wald statistic for the model (12) with $p = 2:$

$$y_{it} = \alpha + \beta_{1t} t + \beta_{2t} t^2 + u_{it}.$$

Thus we have

$$W_p(k) = \frac{1}{\sigma_v^2} \left(\underline{\widehat{\beta}}_{(1k)} - \underline{\widehat{\beta}}_{(2k)} \right)' \left[R(Var(b_a))R' \ \right]^{-1} \left(\underline{\widehat{\beta}}_{(1k)} - \underline{\widehat{\beta}}_{(2k)} \right),$$

where

$$\begin{aligned} b &= \left(\widehat{\beta}_{1(1k)}, \widehat{\beta}_{2(1k)}, \widehat{\beta}_{1(2k)}, \widehat{\beta}_{2(2k)} \right)' \\ &= \left(\underline{\widehat{\beta}}_{(1k)}, \underline{\widehat{\beta}}_{(2k)} \right)' \end{aligned}$$

is a 4×1 vector,

$$Var\,(b_a) = \begin{bmatrix} \left[\sum_{i=1}^{N} \sum_{t=1}^{k} (\underline{t} - \overline{\underline{t}}_{(1k)})'(\underline{t} - \overline{\underline{t}}_{(1k)}) \right]^{-1} & 0_{2\times2} \\ 0_{2\times2} & \left[\sum_{i=1}^{N} \sum_{t=k+1}^{T} (\underline{t} - \overline{\underline{t}}_{(2k)})'(\underline{t} - \overline{\underline{t}}_{(2k)}) \right]^{-1} \end{bmatrix}$$

is a 4×4 matrix, and

$$R = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

is a 2×4 matrix,

$$\underline{t} = \left(t, t^2 \right),$$

$$\underline{\bar{t}}_{(1k)} = \left(\frac{1}{k} \sum_{t=1}^{k} t, \frac{1}{k} \sum_{t=1}^{k} t^2 \right),$$

and

$$\underline{\bar{t}}_{(2k)} = \left(\frac{1}{T-k} \sum_{t=k+1}^{T} t, \frac{1}{T-k} \sum_{t=k+1}^{T} t^2 \right).$$

Define the following test statistics

$$W_{1p}(k) = \frac{\sigma_v^2}{\sigma_0^2} W_p(k),$$

$$W_{2p}(k) = \frac{1}{T^2} \frac{\sigma_v^2}{\sigma_\varepsilon^2} W_p(k),$$

and

$$W_{cp}(k) = \frac{1}{T^2} \frac{\sigma_v^2}{\sigma_\varepsilon^2} W_p(k),$$

Theorem 9 *Under H_0 we have the following:*

1. *If $|\rho| < 1$, then*

$$W_{1p}(k) \xrightarrow{d} P_1(r),$$

2. *If $\rho = 1$, then*

$$W_{2p}(k) \xrightarrow{d} P_2(r),$$

3. *If $\rho = 1 + \frac{c}{T}$, then*

$$W_{cp}(k) \xrightarrow{d} P_c(r),$$

where $P_1(r)$, $P_2(r)$, and $P_c(r)$ are given in the Appendix.

Next we consider the behavior of $W_p(k)$ under sequences of local alternatives. Consider the sequence of local alternatives

$$\beta_{jt}^{(T)} = \beta_j + \frac{1}{T^{(2j-1)/2}} g\left(\frac{t}{T}\right), \text{ for } j = 1, \cdots, p.$$

Theorem 10 *Under the local alternatives (11),* *1. If* $|\rho| < 1$, *then*

$$W_{1p}^{(T)}(k) \xrightarrow{d} P_1(r) + O_p(T),$$

2. If $\rho = 1$, *then*

$$W_{2p}^{(T)}(k) \xrightarrow{d} \left[P_2(r) + \frac{1}{\sigma_\varepsilon^2} R_1(r) \right],$$

3. If $\rho = 1 + \frac{c}{T}$, *then*

$$W_{cp}^{(T)}(k) \xrightarrow{d} \left[P_c(r) + \frac{1}{\sigma_\varepsilon^2} R_1(r) \right],$$

where $P_1(r)$, $P_2(r)$, $P_c(r)$, *and* $R_1(r)$ *are given in the Appendix.*

7 Finite Sample Simulations

First, the critical values for each of the proposed test statistics are simulated. This was accomplished by simulating simple Brownian Motion and then using this to simulate the limiting distributions for each of the proposed test statistics. Each simulation involves $10,000$ replications. The asymptotic critical values of the tests are given in Table 1.

Simulations are also done to examine the empirical size and power of the proposed test statistics. For the size simulations, the model is set as follows:

$$y_{it} = 5 + 2t + \mu_i + v_{it}, i = 1, ..., N, t = 1, ..., T,$$

with $\mu_i \overset{iid}{\sim} N\left(0, \sigma_\mu^2\right)$ and v_{it} follows an AR(1) with

$$v_{it} = \rho v_{it-1} + \varepsilon_{it},$$

$\varepsilon_{it} \overset{iid}{\sim} N\left(0, \sigma_\varepsilon^2\right)$ and $v_{i1} = \sum_{j=0}^{[\kappa T]} \rho^j \varepsilon_{i1-j}$, where $\kappa = 0$ and $\rho = (0, 0.2, 0.4, 0.6, 0.8, 0.9, 0.95, 1.0)$. We fix $\sigma_\mu^2 + \sigma_\varepsilon^2 = 10$ and let $\Delta = \frac{\sigma_\mu^2}{\sigma_\varepsilon^2 + \sigma_\mu^2}$ take the value 0. The following sample size combinations are used for T_1 and T_2: $N = 1, 10, 25, 50$ and $T = 10, 25, 50, 100$. For tests involving W_1 and W_2, the smallest value of T used is 14. This is so that 15% trimming can be used meaningfully. Each experiment involves $5,000$ replications. For each replication we estimated the model using the recursive least squares estimator. Trimming of 15% was used for the tests involving W_1 and W_2. The simulations were performed by an Ultra Enterprise 3000. GAUSS 3.2.31 was used to perform the simulations. Random numbers for μ_i and ε_{it} were generated by the GAUSS procedure RNDNS. At each replication, we generated an $N(T + 1000)$ length of random numbers and then split it into N series so that each series had the same mean and variance. The first $1,000$ observations were discarded for each series.

Tables 2-5 give the empirical sizes of the tests for various values of ρ, N, and T by choosing $\kappa = \Delta = 0$. First we consider T_1, the fluctuations-type test for the case when the disturbances are stationary. Table 2 shows that the empirical size of T_1 is much smaller than the nominal size of five percent when $T = 10$, even when $N = 50$. It can also be seen that the empirical size is essentially zero for $\rho > 0.2$. When the value of T increases, the empirical size increases but is still smaller than the nominal size of five percent but the test has better size properties for the cases with $\rho > 0.2$. When $T = 25$ we see that the empirical size is essentially zero for $\rho > 0.4$. When $T = 50$ we see that the empirical size is essentially zero for $\rho > 0.6$. When $T = 100$ we see that the empirical size is essentially zero for $\rho > 0.8$. From Tables 2-5, we see that the test statistic T_1 has empirical size that is much smaller than the nominal size of five percent in small samples. However, it is important to note that the empirical size does indeed equal the nominal size when the the value of T is very large.

We next consider T_2, the fluctuations-type test for the case when the disturbances are nonstationary, i.e., when $\rho = 1$. Table 2 shows that the empirical size of T_2 is slightly greater than the nominal size of five percent when $T = 10$ and $\rho \geq 0.8$. However, the empirical size is less than the nominal size for $\rho \leq 0.6$. When the value of T increases, the empirical size gets closer to the nominal size of five percent, but only for values of ρ very close to 1. When $T = 25$ the empirical size is essentially zero for $\rho < 0.8$. When $T = 50$ the empirical size is essentially zero for $\rho < 0.9$. When $T = 100$ the empirical size is essentially zero for $\rho < 0.95$. However, if the $\rho = 1$, then T_2 performs quite well, in terms of empirical size, even in small samples. It is also true that if $T = 10$, T_2 outperforms T_1, in terms of empirical size, for all values of $\rho \neq 0$.

Next we consider the Wald-type tests for the case of stationary disturbances, i.e., $\rho < 1$. We consider three statistics, $sup\mathrm{W}_1$, $Mean\mathrm{W}_1$, and $Exp\mathrm{W}_1$. The empirical size of $sup\mathrm{W}_1$ is much smaller than the nominal size in small samples. When $T = 14$ the empirical size of $sup\mathrm{W}_1$ is essentially zero for $\rho > 0.2$. When $T = 25$ the empirical size is essentially zero for $\rho > 0.4$. When $T = 50$ the empirical size is essentially zero for $\rho > 0.6$. When $T = 100$ the empirical size is essentially zero for $\rho > 0.8$. The empirical size of $Mean\mathrm{W}_1$ is slightly larger than the nominal size in small samples when $\rho = 0$. When $T = 14$ the empirical size of $Mean\mathrm{W}_1$ is essentially zero for $\rho > 0.4$. When $T = 25$ the empirical size is essentially zero for $\rho > 0.6$. When $T = 50$ the empirical size is essentially zero for $\rho > 0.8$. When $T = 100$ the empirical size is essentially zero for $\rho > 0.9$. The test statistic $Exp\mathrm{W}_1$ has empirical size approximately twice the nominal size for $\rho = 0$ and $T \leq 25$. Actually, the empirical size of $Exp\mathrm{W}_1$ is larger than the nominal size for $\rho = 0$, in all of the simulations. However, the empirical size is very close to the nominal size of five percent for

$\rho = 0.2$, in all of the simulations. When $T = 14$ the empirical size is essentially zero for $\rho > 0.4$. When $T = 25$ the empirical size is essentially zero for $\rho > 0.6$. When $T = 50$ the empirical size is essentially zero for $\rho > 0.8$. When $T = 100$ the empirical size is essentially zero for $\rho > 0.9$. The statistics $Mean\mathrm{W}_1$ and $Exp\mathrm{W}_1$ have very similar empirical sizes when $T \geq 50$.

We finally consider the Wald-type tests for the case of nonstationary disturbances, i.e., $\rho = 1$. We consider three statistics, $sup\mathrm{W}_2$, $Mean\mathrm{W}_2$, and $Exp\mathrm{W}_2$. The test statistics $sup\mathrm{W}_2$ and $Mean\mathrm{W}_2$ have very similar size properties in the finite sample simulations. When $T = 14$ the empirical size is essentially zero for $\rho < 0.4$. When $T = 25$ the empirical size is essentially zero for $\rho < 0.8$. When $T = 50$ the empirical size is essentially zero for $\rho < 0.9$. When $T = 100$ the empirical size is essentially zero for $\rho < 0.95$. If $\rho = 1$, $sup\mathrm{W}_2$ and $Mean\mathrm{W}_2$ have empirical size approximately equal to the nominal size of five percent, no matter what the sample size. The test statistic $Exp\mathrm{W}_2$ has empirical size equal to 100% in all of the simulations.

The test statistics $Mean\mathrm{W}_1$ and $Exp\mathrm{W}_1$ appear to be superior to T_1 and $sup\mathrm{W}_1$ in small samples, in terms of empirical size. Further, if $T \leq 25$, the test statistic $Mean\mathrm{W}_1$ appears to be superior to $Exp\mathrm{W}_1$. It is also important to note that the empirical size of the test statistics T_1, $sup\mathrm{W}_1$, $Mean\mathrm{W}_1$, and $Exp\mathrm{W}_1$ is approximately equal to the nominal size when the sample size is very large and $\rho = 0$. The test statistics T_2, $sup\mathrm{W}_2$, $Mean\mathrm{W}_2$ are superior to $Exp\mathrm{W}_2$ in small samples, in terms of empirical size. Further, the test statistic T_2 appears to have better empirical size properties for small values of ρ when $T < 25$. It is also important to note that the empirical size of the test statistics T_2, $sup\mathrm{W}_2$, and $Mean\mathrm{W}_2$, and $Exp\mathrm{W}_1$ is approximately equal to the nominal size when the sample size is very large and when $\rho = 1$ for T_2, $sup\mathrm{W}_2$, $Mean\mathrm{W}_2$, and $Exp\mathrm{W}_2$.

To simulate the empirical power of the proposed tests, we generate data from a piecewise trend stationary function. The data generating process is

$$y_{it} = \begin{cases} 5 + 2t + \mu_i + u_{it} & \text{for } t = 1, ..., [\lambda T] \\ \alpha + (2 + 0.2)\, t + \mu_i + u_{it} & \text{for } t = [\lambda T] + 1, ..., T \end{cases}$$

where λ denotes the proportion of the sample at which a structural break occurs. We consider the change points $\lambda = 0.1, 0.2, \cdots, 0.9$.

Tables 6a-6d give the size-corrected power for the test statistic T_1. Several observations are worth mentioning. The size-corrected power of T_1 increases as λ increases or as T increases. Also, for given values of λ and T, the size-corrected power of T_1 increases as N increases. In fact, the size-corrected power is greatly increased for given values of λ, ρ, and T even for the small increase in N from 1 to 10. When $T = 10$ the size-corrected power of T_1 is quite small but begins to increase for $\lambda > 0.6$. It is apparent, even at this small value of T, that the size-corrected power of T_1 increases as N increases. When $T = 100$ and $N = 50$ the size-corrected power of T_1 is essentially 100% when $\lambda > 0.1$ for $\rho = 0$ or 0.2. When $T = 100$ and $N = 50$ the size-corrected power of T_1 is essentially 100% when $\lambda > 0.4$ for all values of $\rho \neq 1$.

Tables 7a-7d give the size-corrected power for the test statistic T_2. Again, several observations are worth mentioning. The size-corrected power of T_2 increases as λ increases or as T increases. Also, for given values of λ and T, the size-corrected power of T_2 increases as N increases. Indeed, the size-corrected power is greatly increased for given values of λ, ρ, and T even for the small increase in N from 1 to 10. When $T = 10$ the size-corrected power of T_2 is quite small but begins to increase when $\lambda > 0.5$. It is apparent, even at this small value of T, that the size-corrected power of T_2 increases as N increases. When $T = 100$ and $N = 50$ the size-corrected power of T_2 is essentially 100% when $\lambda > 0.5$ for all values of ρ.

Actually, all of the test statistics display similar properties concerning size-corrected power. The size-corrected power increases as λ increases or as T increases. Also, for given values of λ and T, the size-corrected power increases as N increases. Indeed, the size-corrected power is greatly increased for given values of λ, ρ, and T even for the small increase in N from 1 to 10.

Tables 8a-8d give the size-corrected power for the test statistic $sup\mathrm{W}_1$. When $T = 100$ and $N = 50$ the size-corrected power of $sup\mathrm{W}_1$ is essentially 100% when $\lambda > 0.1$ for $\rho = 0$, 0.2, 0.4, or 0.6. When $T = 100$ and $N = 50$ the size-corrected power of $sup\mathrm{W}_1$ is essentially 100% when $\lambda > 0.4$ for all values of $\rho \neq 1$. Tables 9a-9d give the size-corrected power for the test statistic $Mean\mathrm{W}_1$. When $T = 100$ and $N = 50$ the size-corrected power of $Mean\mathrm{W}_1$ is essentially 100% when $\lambda > 0.1$ for $\rho = 0$, 0.2, 0.4, or 0.6. When $T = 100$ and $N = 50$ the size-corrected power of $Mean\mathrm{W}_1$ is essentially 100% when $\lambda > 0.4$ for all values of $\rho \neq 1$. Tables 10a-10d give the size-corrected power for the test statistic $Exp\mathrm{W}_1$. When $T = 100$ and $N = 50$ the size-corrected power of $Exp\mathrm{W}_1$ is essentially 100% when $\lambda > 0.1$ for $\rho = 0$, 0.2, 0.4, or 0.6. When $T = 100$ and $N = 50$ the size-corrected power of $Exp\mathrm{W}_1$ is essentially 100% when $\lambda > 0.4$ for all values of $\rho \neq 1$.

The size-corrected power is very similar for each of the statistics, $sup\mathrm{W}_1$, $Mean\mathrm{W}_1$, and $Exp\mathrm{W}_1$, for all values of λ, ρ, N, and T. We can also see that the test statistics $sup\mathrm{W}_1$, $Mean\mathrm{W}_1$, and $Exp\mathrm{W}_1$ all outperform the test statistic T_1, in terms of size-corrected power.

Tables 11a-11d give the size-corrected power for the test statistic $sup\mathrm{W}_2$. When $T = 100$ and $N = 50$ the size-corrected power of $sup\mathrm{W}_2$ is 100% when $\lambda > 0.1$ for $\rho = 0$, 0.2, 0.4, or 0.6. When $T = 100$ and $N = 50$ the size-corrected power of $sup\mathrm{W}_2$ is essentially 100% when $\lambda > 0.5$ for all values of ρ. Tables 12a-12d give the size-corrected power for the test statistic $Mean\mathrm{W}_2$. When $T = 100$ and $N = 50$ the size-corrected power of $Mean\mathrm{W}_2$ is essentially 100% when $\lambda > 0.1$ for $\rho = 0$, 0.2, 0.4, or 0.6. When $T = 100$ and $N = 50$ the size-corrected power of $Mean\mathrm{W}_2$ is essentially 100% when $\lambda > 0.5$ for all values of ρ. Tables 13a-13d give the size-corrected power for the test statistic $Exp\mathrm{W}_2$. When $T = 100$ and $N = 50$ the size-corrected power of $Exp\mathrm{W}_2$ is essentially 100% when $\lambda > 0.2$ for $\rho = 0$, 0.2, 0.4, 0.6 or 0.8. When $T = 100$ and $N = 50$ the size-corrected power of $Exp\mathrm{W}_2$ is essentially 100% when $\lambda > 0.5$ for all values of ρ.

The size-corrected power is also very similar for each of the statistics, $sup\mathrm{W}_2$, $Mean\mathrm{W}_2$, and $Exp\mathrm{W}_2$, for all values of λ, ρ, N, and T. Additionally, we see that the test statistics $sup\mathrm{W}_2$, $Mean\mathrm{W}_2$, and $Exp\mathrm{W}_2$ are very similar to the test statistic T_2 in terms of size-corrected power.

The results for the empirical size of the test statistics and for the size-corrected power of the test statistics are very similar for the cases where nominal size is ten percent and one percent. Hence, only the cases with nominal size equal to five percent are reported here.

8 Conclusion

This paper proposes a class of tests for testing the structural change of a time trend model in panel data. The results are confirmed by means of Monte Carlo experiments. The simulations show that the sample sizes must be quite large in order for the test statistics to have optimal size and power properties when $\rho < 1$. The test statistic $Mean\mathrm{W}_1$ is the preferred test statistic when the disturbances are stationary, i.e., when $\rho < 1$. For the case when the disturbances are nonstationary, the test statistics T_2, $sup\mathrm{W}_2$, and $Mean\mathrm{W}_2$ are all preferred to $Exp\mathrm{W}_2$. However, both the empirical size and size-corrected power properties of all of the test statistics are greatly improved as N increases, for any value of T. In very small samples, it may be necessary to use finite sample critical values instead of the critical values obtained from the limiting distributions. Using this technique will result in optimal size properties and the power properties will be as reported in Tables 6a-13d.

Appendix

A Proof of Theorem 1

Proof. Note that $\widehat{\beta}_k - \widehat{\beta}_T$ can be written as

$$\widehat{\beta}_k - \widehat{\beta}_T = \left(\widehat{\beta}_k - \beta\right) - \left(\widehat{\beta}_T - \beta\right).$$

Under H_0 we know

$$\begin{aligned}
\sqrt{NT^3}\left(\widehat{\beta}_k - \beta\right) &= \frac{\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[\frac{1}{T^{3/2}}\sum_{t=1}^{k}(t-\bar{t}_k)u_{it}\right]}{\frac{1}{NT^3}N\sum_{t=1}^{k}(t-\bar{t}_k)^2} \\
&= \frac{\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[\frac{1}{T^{3/2}}\sum_{t=1}^{k}(t-\bar{t}_k)v_{it}\right]}{\frac{1}{T^3}\sum_{t=1}^{k}(t-\bar{t}_k)^2}.
\end{aligned}$$

Note

$$\frac{1}{T^3}\sum_{t=1}^{k}(t-\bar{t}_k)^2 = \frac{k^3}{T^3}\frac{1}{k^3}\sum_{t=1}^{k}(t-\bar{t}_k)^2 \to \frac{1}{12}r^3$$

as $T \to \infty$ and

$$\begin{aligned}
\frac{1}{T^{3/2}}\sum_{t=1}^{k}(t-\bar{t}_k)v_{it} &= \frac{1}{T^{3/2}}\sum_{t=1}^{k}(t-\frac{1}{k}\sum_{t=1}^{k}t)v_{it} \\
&= \frac{1}{T^{3/2}}\sum_{t=1}^{k}(t-\frac{k\,(k+1)}{2k})v_{it} \\
&= \frac{1}{T^{3/2}}\left[\sum_{t=1}^{k}tv_{it} - \frac{k+1}{2}\left(\sum_{t=1}^{k}v_{it}\right)\right] \\
&= \frac{1}{T^{3/2}}\left[\sum_{t=1}^{k}tv_{it} - \frac{k}{2}\left(\sum_{t=1}^{k}v_{it}\right) - \frac{1}{2}\sum_{t=1}^{k}v_{it}\right] \\
&\xrightarrow{d} \sigma_0\left[rW_i(r) - \int_0^r W_i(s)ds\right] - \frac{1}{2}\sigma_0 rW_i(r) \\
&= \frac{1}{2}\sigma_0\left[rW_i(r) - 2\int_0^r W_i(r)dr\right]
\end{aligned}$$

uniformly in r as $T \to \infty$ since (e.g., Lemma 1.1, Chu and White, 1992)

$$\frac{1}{T^{3/2}}\sum_{t=1}^{k}tv_{it} \xrightarrow{d} \sigma_0\left[rW_i(r) - \int_0^r W_i(s)ds\right],$$

$$\frac{1}{T^{3/2}}\frac{k}{2}\left(\sum_{t=1}^{k}v_{it}\right) = \frac{1}{2T^{1/2}}\frac{k}{T}\left(\sum_{t=1}^{k}v_{it}\right) \xrightarrow{d} \frac{1}{2}\sigma_0 rW_i(r),$$

uniformly in r and

$$\frac{1}{T^{3/2}}\frac{1}{2}\sum_{t=1}^{k}v_{it} = o_p(1)$$

where

$$\sigma_0^2 = \sigma_\varepsilon^2\left(\frac{1}{1-\rho}\right)^2.$$

Now

$$
\begin{aligned}
& \sqrt{NT}\left(\widehat{\beta}_k-\beta\right) \\
= \quad & \frac{\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[\frac{1}{T^{3/2}}\sum_{t=1}^{k}(t-\bar{t}_k)v_{it}\right]}{\frac{1}{T^3}\sum_{t=1}^{k}(t-\bar{t}_k)^2} \\
= \quad & 12\left(\frac{k}{T}\right)^{-3}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[\frac{1}{T^{3/2}}\sum_{t=1}^{k}(t-\bar{t}_k)v_{it}\right]+o_p(1).
\end{aligned}
$$

For a fixed N as $T\to\infty$ we have

$$
12\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T^{3/2}}\sum_{t=1}^{k}(t-\bar{t}_k)v_{it}\xrightarrow{d}\frac{6\sigma_0}{\sqrt{N}}\sum_{i=1}^{N}\left[rW_i(r)-2\int_0^r W_i(s)ds\right]=\frac{6\sigma_0}{\sqrt{N}}\sum_{i=1}^{N}G_i(r)
$$

uniformly in r, where $G_i(r)=rW_i(r)-2\int_0^r W_i(s)ds$ is a Gaussian process with zero mean and variance, $\frac{1}{3}r^3$, i.e., for each r

$$
G_i(r)\sim \mathrm{N}(0,\frac{1}{3}r^3).
$$

Hence

$$
\frac{1}{\sqrt{N}}\sum_{i=1}^{N}G_i(r)\sim \mathrm{N}(0,\frac{1}{3}r^3)
$$

for all N. Finally,

$$
\sqrt{NT^3}\frac{1}{6\sigma_0}\left(\frac{k}{T}\right)^3\left(\widehat{\beta}_k-\beta\right)\xrightarrow{d}G(r).
$$

Similarly,

$$
\sqrt{NT^3}\frac{1}{6\sigma_0}\left(\widehat{\beta}_T-\beta\right)\xrightarrow{d}G(1).
$$

It follows that

$$
\begin{aligned}
\sqrt{NT^3}\frac{1}{6\sigma_0}\left(\frac{k}{T}\right)^3\left(\widehat{\beta}_k-\widehat{\beta}_T\right) &= \sqrt{NT^3}\frac{1}{6\sigma_0}\left(\frac{k}{T}\right)^3\left(\widehat{\beta}_k-\beta\right)-\sqrt{NT^3}\left(\frac{k}{T}\right)^3\frac{1}{6\sigma_0}\left(\widehat{\beta}_T-\beta\right) \\
\xrightarrow{d} G(r)-r^3G(1) &= G_0(r)
\end{aligned}
$$

uniformly in r as T$\to\infty$ for all N proving Theorem 1. ■

B Proof of Theorem 2

Proof. First we note that

$$
\left(\widehat{\beta}_{1k}-\widehat{\beta}_{2k}\right)=\left(\widehat{\beta}_{1k}-\beta\right)-\left(\widehat{\beta}_{2k}-\beta\right).
$$

Then under H_0

$$
\begin{aligned}
&\sqrt{NT^3}\left(\widehat{\beta}_{1k}-\widehat{\beta}_{2k}\right) && (13)\\
=\ & \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{NT^3}\left(\widehat{\beta}_{1k}-\beta\right) \\ \sqrt{NT^3}\left(\widehat{\beta}_{2k}-\beta\right) \end{bmatrix} \\
=\ & \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[\frac{1}{T^{3/2}}\sum_{t=1}^{k}(t-\bar{t}_{1k})v_{it}\right]}{\frac{1}{T^{t}}\sum_{t=1}^{k}(t-\bar{t}_{1k})^2} \\ \frac{\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[\frac{1}{T^{3/2}}\sum_{t=k+1}^{T}(t-\bar{t}_{2k})v_{it}\right]}{\frac{1}{T^{3}}\sum_{t=k+1}^{T}(t-\bar{t}_{2k})^2} \end{bmatrix} && (14)\\
=\ & \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 12r^{-3}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[\frac{1}{T^{3/2}}\sum_{t=1}^{k}(t-\bar{t}_{1k})v_{it}\right] \\ 12\left(1-r\right)^{-3}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[\frac{1}{T^{3/2}}\sum_{t=k+1}^{T}(t-\bar{t}_{2k})v_{it}\right] \end{bmatrix} + o_p(1) \\
\xrightarrow{d}\ & \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 6\sigma_0 r^{-3}G(r) \\ 6\sigma_0\left(1-r\right)^{-3}\left(G(1)-G(r)+W(r)-rW(1)\right) \end{bmatrix} \\
=\ & 6\sigma_0\left[\frac{G(r)}{r^3}-\frac{G(1)-G(r)+W(r)-rW(1)}{\left(1-r\right)^3}\right] && (15)
\end{aligned}
$$

uniformly in r since

$$
\begin{aligned}
\frac{1}{T^3}\sum_{t=k+1}^{T}(t-\bar{t}_{2k})^2 &= \frac{\left(T-k\right)^3}{T^3}\frac{1}{\left(T-k\right)^3}\sum_{t=k+1}^{T}(t-\bar{t}_{2k})^2 \\
&\to \frac{1}{12}\left(1-r\right)^3,
\end{aligned}
$$

and

$$
\begin{aligned}
&\begin{bmatrix} 12r^{-3}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[\frac{1}{T^{3/2}}\sum_{t=1}^{k}(t-\bar{t}_{1k})v_{it}\right] \\ 12\left(1-r\right)^{-3}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[\frac{1}{T^{3/2}}\sum_{t=k+1}^{T}(t-\bar{t}_{2k})v_{it}\right] \end{bmatrix} \\
&\xrightarrow{d}. \begin{bmatrix} 6\sigma_0 r^{-3}G(r) \\ 6\sigma_0\left(1-r\right)^{-3}\left(G(1)-G(r)+W(r)-rW(1)\right) \end{bmatrix}
\end{aligned}
$$

uniformly in r as $T\to\infty$ for all N. Note

$$
\begin{aligned}
&\left(\frac{1}{NT^3}\sum_{i=1}^{N}\sum_{t=1}^{k}(t-\bar{t}_{1k})^2\right)^{-1}+\left(\frac{1}{NT^3}\sum_{i=1}^{N}\sum_{t=k+1}^{T}(t-\bar{t}_{2k})^2\right)^{-1} \\
=\ & \left(\frac{1}{T^3}\sum_{t=1}^{k}(t-\bar{t}_{1k})^2\right)^{-1}+\left(\frac{1}{T^3}\sum_{t=k+1}^{T}(t-\bar{t}_{2k})^2\right)^{-1} \\
\to\ & \left(\frac{1}{12}r^3\right)^{-1}+\left(\frac{1}{12}\left(1-r\right)^3\right)^{-1} \\
=\ & \frac{12}{r^3}+\frac{12}{\left(1-r\right)^3}. && (16)
\end{aligned}
$$

Then

$$\begin{aligned}
\mathrm{W}_1(k) &= \frac{1}{3\sigma_0^2}\frac{\left[\sqrt{NT^3}\left(\widehat{\beta}_{1k}-\widehat{\beta}_{2k}\right)\right]^2}{\left[\left(\frac{1}{NT^3}\sum_{i=1}^N\sum_{t=1}^k(t-\bar{t}_{1k})^2\right)^{-1}+\left(\frac{1}{NT^3}\sum_{i=1}^N\sum_{t=k+1}^T(t-\bar{t}_{2k})^2\right)^{-1}\right]} \\
&\xrightarrow{d} \frac{1}{3\sigma_0^2}\frac{\left\{6\sigma_0\left[\frac{G(r)}{r^3}-\frac{G(1)-G(r)+W(r)-rW(1)}{(1-r)^3}\right]\right\}^2}{\frac{12}{r^3}+\frac{12}{(1-r)^3}} \\
&= \frac{\left[\frac{G(r)}{r^3}-\frac{G(1)-G(r)+W(r)-rW(1)}{(1-r)^3}\right]^2}{\frac{1}{r^3}+\frac{1}{(1-r)^3}} \\
&= \frac{\left[\frac{G(r)(1-r)^3-[G(1)-G(r)+W(r)-rW(1)]r^3}{r^3(1-r)^3}\right]^2}{\frac{(1-r)^3+r^3}{r^3(1-r)^3}} \\
&= \frac{\left[G(r)(1-r)^3-r^3\left[G(1)-G(r)+W(r)-rW(1)\right]\right]^2}{r^3(1-r)^3\left[(1-r)^3+r^3\right]} \\
&= \left\{\frac{G(r)(1-r)^3-r^3\left[G(1)-G(r)+W(r)-rW(1)\right]}{\left[r^3(1-r)^3\left[(1-r)^3+r^3\right]\right]^{\frac{1}{2}}}\right\}^2
\end{aligned}$$

uniformly in r proving Theorem 2. ■

C Proof of Theorem 3

Proof. Note that $v_{it}=\sum_{j=0}^t\varepsilon_{ij}$ so

$$\begin{aligned}
\frac{1}{T^{5/2}}\sum_{t=1}^k(t-\bar{t}_{1k})u_{it} &= \frac{1}{T^{5/2}}\sum_{t=1}^k(t-\bar{t}_{1k})v_{it} \\
&= \frac{1}{T^{5/2}}\sum_{t=1}^k(t-\frac{1}{k}\sum_{t=1}^k t)v_{it} \\
&= \frac{1}{T^{5/2}}\left[\sum_{t=1}^k tv_{it}-\frac{(k+1)}{2}\left(\sum_{t=1}^k v_{it}\right)\right] \\
&= \frac{1}{T^{5/2}}\left[\sum_{t=1}^k tv_{it}-\frac{k}{2}\left(\sum_{t=1}^k v_{it}\right)-\frac{1}{2}\sum_{t=1}^k v_{it}\right] \\
&\xrightarrow{d} \sigma_\varepsilon\int_0^r s\left[W(s)+\widetilde{W}(\kappa)\right]ds-\frac{\sigma_\varepsilon}{2}r\int_0^r\left[W(s)+\widetilde{W}(\kappa)\right]ds \\
&= \frac{\sigma_\varepsilon}{2}\left[2\int_0^r s\left[W(s)+\widetilde{W}(\kappa)\right]ds-r\int_0^r\left[W(s)+\widetilde{W}(\kappa)\right]ds\right]
\end{aligned}$$

since

$$\frac{1}{T^{5/2}}\sum_{t=1}^{k} tv_{it} \xrightarrow{d} \sigma_\varepsilon \int_0^r s\left[W(s)+\widetilde{W}(\kappa)\right]ds,$$

$$\begin{aligned}\frac{1}{T^{5/2}}\frac{k}{2}\sum_{t=1}^{k} v_{it} &= \frac{1}{2}\frac{k}{T}\frac{1}{T^{3/2}}\sum_{t=1}^{k} v_{it} \\ &\xrightarrow{d} \frac{\sigma_\varepsilon}{2} r\int_0^r \left[W(s)+\widetilde{W}(\kappa)\right]ds,\end{aligned}$$

and

$$\frac{1}{T^{5/2}}\sum_{t=1}^{k} v_{it} = o_p(1)$$

uniformly in r. Then

$$\begin{aligned}\sqrt{NT}\left(\widehat{\beta}_k-\beta\right) &= \frac{\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[\frac{1}{T^{5/2}}\sum_{t=1}^{k}(t-\bar{t}_{1k})u_{it}\right]}{\frac{1}{NT^3}N\sum_{t=1}^{k}(t-\bar{t}_{1k})^2} \\ &= 12\left(\frac{k}{T}\right)^{-3}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[\frac{1}{T^{5/2}}\sum_{t=1}^{k}(t-\bar{t}_{1k})v_{it}\right]+o_p(1) \\ &\xrightarrow{d} 12r^{-3}\frac{\sigma_\varepsilon}{2}\left[2\int_0^r s\left[W(s)+\widetilde{W}(\kappa)\right]ds - r\int_0^r \left[W(s)+\widetilde{W}(\kappa)\right]ds\right] \\ &= 6\sigma_\varepsilon r^{-3}\left[2\int_0^r s\left[W(s)+\widetilde{W}(\kappa)\right]ds - r\int_0^r \left[W(s)+\widetilde{W}(\kappa)\right]ds\right].\end{aligned}$$

uniformly in r as $T\to\infty$ for all N. It follows that

$$\sqrt{NT}\frac{1}{6\sigma_\varepsilon}\left(\frac{k}{T}\right)^3\left(\widehat{\beta}_k-\beta\right) \xrightarrow{d} 2\int_0^r s\left[W(s)+\widetilde{W}(\kappa)\right]ds - r\int_0^r \left[W(s)+\widetilde{W}(\kappa)\right]ds = H(r).$$

In a similar fashion, we have

$$\sqrt{NT}\frac{1}{6\sigma_\varepsilon}\left(\widehat{\beta}_T-\beta\right) \xrightarrow{d} 2\int_0^1 s\left[W(s)+\widetilde{W}(\kappa)\right]ds - \int_0^1 \left[W(s)+\widetilde{W}(\kappa)\right]ds = H(1).$$

uniformly in r as $T\to\infty$ for all N. We therefore prove the Theorem 3, i.e.,

$$\sqrt{NT}\frac{1}{6\sigma_\varepsilon}\left(\frac{k}{T}\right)^3\left(\widehat{\beta}_k-\widehat{\beta}_T\right) \xrightarrow{d} H(r)-r^3H(1)=H_0(r).$$

uniformly in r as $T\to\infty$ for all N. ■

D Proof of Theorem 4

Proof. From the proof of Theorem 3 we know that

$$\sqrt{NT}\left(\widehat{\beta}_{1k}-\beta\right)\xrightarrow{d}6\sigma_{\varepsilon}r^{-3}H(r)$$

and

$$\sqrt{NT}\left(\widehat{\beta}_{2k}-\beta\right)\xrightarrow{d}6\sigma_{\varepsilon}\left(1-r\right)^{-3}\left[H(1)-H(r)-r\int_0^1 W(s)ds+\int_0^r W(s)ds\right]$$

uniformly in r. Then

$$\begin{aligned}
&\sqrt{NT}\left(\widehat{\beta}_{1k}-\widehat{\beta}_{2k}\right)\\
=\;&\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}\sqrt{NT}\left(\widehat{\beta}_{1k}-\beta\right)\\ \sqrt{NT}\left(\widehat{\beta}_{2k}-\beta\right)\end{bmatrix}\\
\xrightarrow{d}\;&\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}6\sigma_{\varepsilon}r^{-3}H(r)\\ 6\sigma_{\varepsilon}\left(1-r\right)^{-3}\begin{pmatrix}H(1)-H(r)-\\ r\int_0^1 W(s)ds+\int_0^r W(s)ds\end{pmatrix}\end{bmatrix}+o_p(1)\\
=\;&6\sigma_{\varepsilon}\left[\frac{H(r)}{r^3}-\frac{H(1)-H(r)-r\int_0^1 W(s)ds+\int_0^r W(s)ds}{\left(1-r\right)^3}\right]
\end{aligned}$$

uniformly in r. Also

$$\begin{aligned}
&\left(\frac{1}{NT^3}\sum_{i=1}^{N}\sum_{t=1}^{k}(t-\overline{t}_{1k})^2\right)^{-1}+\left(\frac{1}{NT^3}\sum_{i=1}^{N}\sum_{t=k+1}^{T}(t-\overline{t}_{2k})^2\right)^{-1}\\
\rightarrow\;&\frac{12}{r^3}+\frac{12}{\left(1-r\right)^3}
\end{aligned}$$

uniformly in r. Hence

$$\begin{aligned}
\frac{1}{T^2}\mathrm{W}_2(k) &= \frac{1}{3\sigma_{\varepsilon}^2}\frac{\left[\sqrt{NT}\left(\widehat{\beta}_{1k}-\widehat{\beta}_{2k}\right)\right]^2}{\left[\left(\frac{1}{NT^3}\sum_{i=1}^{N}\sum_{t=1}^{k}(t-\overline{t}_{1k})^2\right)^{-1}+\left(\frac{1}{NT^3}\sum_{i=1}^{N}\sum_{t=k+1}^{T}(t-\overline{t}_{2k})^2\right)^{-1}\right]}\\
&\xrightarrow{d}\frac{1}{3\sigma_{\varepsilon}^2}\frac{\left\{6\sigma_{\varepsilon}\left(\frac{H(r)}{r^3}-\frac{H(1)-H(r)-r\int_0^1 W(s)ds+\int_0^r W(s)ds}{(1-r)^3}\right)\right\}^2}{\frac{12}{r^3}+\frac{12}{(1-r)^3}}\\
&=\left\{\frac{H(r)\left(1-r\right)^3-\left(H(1)-H(r)-r\int_0^1 W(s)ds+\int_0^r W(s)ds\right)r^3}{\left(r^3\left(1-r\right)^3\left(\left(1-r\right)^3+r^3\right)\right)^{1/2}}\right\}^2
\end{aligned}$$

uniformly in r proving Theorem 4. ■

E Proof of Theorem 5

Proof. Note that $v_{it} = \rho v_{it-1} + \varepsilon_{it}$ with $\rho = 1 + c/T$, so

$$\begin{aligned}
\frac{1}{T^{5/2}}\sum_{t=1}^{k}(t-\bar{t}_{1k})u_{it} &= \frac{1}{T^{5/2}}\sum_{t=1}^{k}(t-\bar{t}_{1k})v_{it} \\
&= \frac{1}{T^{5/2}}\sum_{t=1}^{k}(t-\frac{1}{k}\sum_{t=1}^{k}t)v_{it} \\
&= \frac{1}{T^{5/2}}\left[\sum_{t=1}^{k}tv_{it}-\frac{(k+1)}{2}\left(\sum_{t=1}^{k}v_{it}\right)\right] \\
&= \frac{1}{T^{5/2}}\left[\sum_{t=1}^{k}tv_{it}-\frac{k}{2}\left(\sum_{t=1}^{k}v_{it}\right)-\frac{1}{2}\sum_{t=1}^{k}v_{it}\right] \\
&\xrightarrow{d} \sigma_\varepsilon\int_0^r s\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds-\frac{\sigma_\varepsilon}{2}r\int_0^r\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds \\
&= \frac{\sigma_\varepsilon}{2}\left[2\int_0^r s\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds-r\int_0^r\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds\right]
\end{aligned}$$

uniformly in r since

$$\frac{1}{T^{5/2}}\sum_{t=1}^{k}tv_{it}\xrightarrow{d}\sigma_\varepsilon\int_0^r s\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds,$$

$$\begin{aligned}
\frac{1}{T^{5/2}}\frac{k}{2}\sum_{t=1}^{k}v_{it} &= \frac{1}{2}\frac{k}{T}\frac{1}{T^{3/2}}\sum_{t=1}^{k}v_{it} \\
&\xrightarrow{d}\frac{\sigma_\varepsilon}{2}r\int_0^r\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds,
\end{aligned}$$

and

$$\frac{1}{T^{5/2}}\sum_{t=1}^{k}v_{it}=o_p(1)$$

uniformly in r. Then

$$\begin{aligned}
\sqrt{NT}\left(\widehat{\beta}_k-\beta\right) &= \frac{\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[\frac{1}{T^{5/2}}\sum_{t=1}^{k}(t-\bar{t}_{1k})u_{it}\right]}{\frac{1}{NT^3}N\sum_{t=1}^{k}(t-\bar{t}_{1k})^2} \\
&= 12\left(\frac{k}{T}\right)^{-3}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[\frac{1}{T^{5/2}}\sum_{t=1}^{k}(t-\bar{t}_{1k})v_{it}\right]+o_p(1) \\
&\xrightarrow{d} 12r^{-3}\frac{\sigma_\varepsilon}{2}\left[2\int_0^r s\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds-r\int_0^r\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds\right] \\
&= 6\sigma_\varepsilon r^{-3}\left[2\int_0^r s\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds-r\int_0^r\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds\right].
\end{aligned}$$

uniformly in r as $T \to \infty$ for all N. It follows that

$$\sqrt{NT}\frac{1}{6\sigma_\varepsilon}\left(\frac{k}{T}\right)^3\left(\widehat{\beta}_k-\beta\right)\xrightarrow{d}2\int_0^r s\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds-r\int_0^r\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds=H_c(r).$$

In a similar fashion, we have

$$\sqrt{NT}\frac{1}{6\sigma_\varepsilon}\left(\widehat{\beta}_T-\beta\right)\xrightarrow{d}2\int_0^1 s\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds-\int_0^1\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds=H_c(1).$$

uniformly in r as $T \to \infty$ for all N. We therefore prove Theorem 7, i.e.,

$$\sqrt{NT}\frac{1}{6\sigma_\varepsilon}\left(\frac{k}{T}\right)^3\left(\widehat{\beta}_k-\widehat{\beta}_T\right)\xrightarrow{d}H_c(r)-r^3H_c(1).$$

uniformly in r as $T \to \infty$ for all N. ■

F Proof of Theorem 6

Proof. From the proof of Theorem 5 we know that

$$\sqrt{NT}\left(\widehat{\beta}_{1k}-\beta\right)\xrightarrow{d}6\sigma_\varepsilon r^{-3}H_c(r)$$

and

$$\begin{aligned}&\sqrt{NT}\left(\widehat{\beta}_{2k}-\beta\right)\\&\xrightarrow{d}6\sigma_\varepsilon\left(1-r\right)^{-3}\left[H_c(1)-H_c(r)-r\int_0^1\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds+\int_0^r\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds\right]\end{aligned}$$

uniformly in r. Then

$$\begin{aligned}&\sqrt{NT}\left(\widehat{\beta}_{1k}-\widehat{\beta}_{2k}\right)\\=&\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}\sqrt{NT}\left(\widehat{\beta}_{1k}-\beta\right)\\\sqrt{NT}\left(\widehat{\beta}_{2k}-\beta\right)\end{bmatrix}\\&\xrightarrow{d}\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}6\sigma_\varepsilon r^{-3}H_c(r)\\6\sigma_\varepsilon\left(1-r\right)^{-3}\begin{pmatrix}H_c(1)-H_c(r)-\\r\int_0^1\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds\\+\int_0^r\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds\end{pmatrix}\end{bmatrix}\\&+o_p(1)\\=&6\sigma_\varepsilon\left[\frac{H_c(r)}{r^3}-\frac{\left[H_c(1)-H_c(r)-r\int_0^1\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds+\int_0^r\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds\right]}{\left(1-r\right)^3}\right]\end{aligned}$$

uniformly in r. Also

$$
\begin{aligned}
&\left(\frac{1}{NT^3}\sum_{i=1}^{N}\sum_{t=1}^{k}(t-\overline{t}_{1k})^2\right)^{-1}+\left(\frac{1}{NT^3}\sum_{i=1}^{N}\sum_{t=k+1}^{T}(t-\overline{t}_{2k})^2\right)^{-1} \\
\rightarrow\quad & \frac{12}{r^3}+\frac{12}{(1-r)^3}.
\end{aligned}
$$

uniformly in r. Hence

$$
\begin{aligned}
& \frac{1}{T^2}\mathrm{W}_c(k) \\
= \quad & \frac{1}{3\sigma_\varepsilon^2}\frac{\left[\sqrt{NT}\left(\widehat{\beta}_{1k}-\widehat{\beta}_{2k}\right)\right]^2}{\left[\left(\frac{1}{NT^3}\sum_{i=1}^{N}\sum_{t=1}^{k}(t-\overline{t}_{1k})^2\right)^{-1}+\left(\frac{1}{NT^3}\sum_{i=1}^{N}\sum_{t=k+1}^{T}(t-\overline{t}_{2k})^2\right)^{-1}\right]} \\
\xrightarrow{d} \quad & \frac{1}{3\sigma_\varepsilon^2}\frac{\left\{6\sigma_\varepsilon\left[\frac{H_c(r)}{r^3}-\frac{\left\{H_c(1)-H_c(r)-r\int_0^1\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds+\int_0^r\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds\right\}}{(1-r)^3}\right]\right\}^2}{\frac{12}{r^3}+\frac{12}{(1-r)^3}} \\
= \quad & \left\{\frac{H_c(r)\,(1-r)^3-\left(\begin{array}{c}H_c(1)-H_c(r)-r\int_0^1\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds+\\ \int_0^r\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds\end{array}\right)r^3}{\left[r^3\,(1-r)^3\left[(1-r)^3+r^3\right]\right]^{1/2}}\right\}^2
\end{aligned}
$$

uniformly in r proving Theorem 6. ■

G Proof of Theorem 7

Proof. The model under the alternative is

$$
\begin{aligned}
y_{it}^{(T)} &= \alpha+\beta_t^{(T)}t+u_{it} \\
&= \alpha+\beta t+\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)t+u_{it}.
\end{aligned}
$$

Note

$$
\begin{aligned}
\widehat{\beta}_k^{(T)} &= \frac{\sum_{i=1}^N \sum_{t=1}^k (t-\bar{t}_{1k}) y_{it}^{(T)}}{N\sum_{t=1}^k (t-\bar{t}_{1k})^2} \\
&= \frac{\sum_{i=1}^N \sum_{t=1}^k (t-\bar{t}_{1k}) \left[\alpha+\beta t+\frac{1}{\sqrt{T}} g\left(\frac{t}{T}\right) t+u_{it}\right]}{N\sum_{t=1}^k (t-\bar{t}_{1k})^2} \\
&= \beta+\frac{\sum_{i=1}^N \sum_{t=1}^k (t-\bar{t}_{1k}) \left[\frac{1}{\sqrt{T}} g\left(\frac{t}{T}\right) t+u_{it}\right]}{N\sum_{t=1}^k (t-\bar{t}_{1k})^2} \\
&= \beta+\frac{\sum_{i=1}^N \sum_{t=1}^k (t-\bar{t}_{1k}) t\frac{1}{\sqrt{T}} g\left(\frac{t}{T}\right)}{N\sum_{t=1}^k (t-\bar{t}_{1k})^2}+\frac{\sum_{i=1}^N \sum_{t=1}^k (t-\bar{t}_{1k}) v_{it}}{N\sum_{t=1}^k (t-\bar{t}_{1k})^2}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\sqrt{NT^3}\left(\widehat{\beta}_k^{(T)}-\beta\right) \\
&= \frac{\frac{1}{\sqrt{N}}\sum_{i=1}^N \frac{1}{T^{3/2}}\sum_{t=1}^k (t-\bar{t}_{1k}) t\frac{1}{\sqrt{T}} g\left(\frac{t}{T}\right)}{\frac{1}{NT^3}N\sum_{t=1}^k (t-\bar{t}_{1k})^2}+\frac{\frac{1}{\sqrt{N}}\sum_{i=1}^N \frac{1}{T^{3/2}}\sum_{t=1}^k (t-\bar{t}_{1k}) v_{it}}{\frac{1}{NT^3}N\sum_{t=1}^k (t-\bar{t}_{1k})^2} \\
&= 12\left(\frac{k}{T}\right)^{-3}\left[\frac{1}{\sqrt{N}}\sum_{i=1}^N \frac{1}{T^{3/2}}\sum_{t=1}^k (t-\bar{t}_{1k}) t\frac{1}{\sqrt{T}} g\left(\frac{t}{T}\right)+\frac{1}{\sqrt{N}}\sum_{i=1}^N \frac{1}{T^{3/2}}\sum_{t=1}^k (t-\bar{t}_{1k}) v_{it}\right]+o_p(1)
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&\sqrt{NT^3}\left(\widehat{\beta}_T^{(T)}-\beta\right) \\
&= 12\frac{1}{\sqrt{N}}\sum_{i=1}^N \frac{1}{T^{3/2}}\sum_{t=1}^T (t-\bar{t}) t\frac{1}{\sqrt{T}} g\left(\frac{t}{T}\right)+\frac{1}{\sqrt{N}}\sum_{i=1}^N \frac{1}{T^{3/2}}\sum_{t=1}^T (t-\bar{t}) v_{it}+o_p\left(1\right)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\sqrt{NT^3}\left(\widehat{\beta}_k^{(T)}-\widehat{\beta}_T^{(T)}\right) \\
&= \frac{1}{\sqrt{N}}\sum_{i=1}^N \left[12\left(\frac{k}{T}\right)^{-3}\frac{1}{T^{3/2}}\sum_{t=1}^k (t-\bar{t}_{1k}) v_{it}-\frac{12}{T^{3/2}}\sum_{t=1}^T (t-\bar{t}) v_{it}\right] \\
&\quad +\frac{1}{\sqrt{N}}\sum_{i=1}^N \left[12\left(\frac{k}{T}\right)^{-3}\frac{1}{T^{3/2}}\sum_{t=1}^k (t-\bar{t}_{1k}) t\frac{1}{\sqrt{T}} g\left(\frac{t}{T}\right)-\frac{12}{\sqrt{T}}\sum_{t=1}^T (t-\bar{t}) t\frac{1}{\sqrt{T}} g\left(\frac{t}{T}\right)\right]+o_p\left(1\right)
\end{aligned}
$$

From Theorem 1 we know that

$$
\frac{1}{6\sigma_0}\left[\frac{1}{T^{3/2}}\sum_{t=1}^k (t-\bar{t}_{1k}) v_{it}-\left(\frac{k}{T}\right)^3\frac{1}{T^{3/2}}\sum_{t=1}^T (t-\bar{t}) v_{it}\right]\xrightarrow{d} G_0(r)
$$

uniformly in r. It is easy to show that

$$
\left[\frac{1}{T^{3/2}}\sum_{t=1}^k (t-\bar{t}_{1k}) t\frac{1}{\sqrt{T}} g\left(\frac{t}{T}\right)-\left(\frac{k}{T}\right)^3\frac{1}{T^{3/2}}\sum_{t=1}^T (t-\bar{t}) t\frac{1}{\sqrt{T}} g\left(\frac{t}{T}\right)\right]=O_p\left(T\right)
$$

uniformly in r since (e.g., Bai, 1996, p. 609)

$$\begin{aligned} & \frac{1}{T^{5/2}}\sum_{t=1}^{k}(t-\bar{t}_{1k})t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right) \\ \rightarrow\; & \int_0^r \frac{d\frac{1}{3}s^3}{ds}g(s)ds-\frac{1}{2}r\int_0^r \frac{d\frac{1}{2}s^2}{ds}g(s)ds \\ =\; & \int_0^r s^2 g(s)ds-\frac{1}{2}r\int_0^r sg(s)ds \end{aligned}$$

and

$$\frac{1}{T^{5/2}}\sum_{t=1}^{T}(t-\bar{t})t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)\rightarrow\int_0^1 s^2g(s)ds-\frac{1}{2}\int_0^1 sg(s)ds$$

uniformly in s. Therefore, for a fixed N

$$\begin{aligned} & \sqrt{NT^3}\frac{1}{6\sigma_0}\left(\frac{k}{T}\right)^3\left(\hat{\beta}_k^{(T)}-\hat{\beta}_T^{(T)}\right) \\ & \xrightarrow{d}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\{G_0(r)+O_p(T)\} \end{aligned}$$

uniformly in r as $T\rightarrow\infty$. Then

$$\sqrt{NT^3}\frac{1}{6\sigma_0}\left(\frac{k}{T}\right)^3\left(\widehat{\beta}_k^{(T)}-\widehat{\beta}_T^{(T)}\right)\xrightarrow{d}G_0(r)+O_p(T)$$

uniformly in r for all N proving (a).

The proofs of (b) and (c) are similar to (a) with a different speed.

$$\begin{aligned} & \sqrt{NT}\left(\widehat{\beta}_k^{(T)}-\beta\right) \\ =\; & \frac{\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T^{5/2}}\sum_{t=1}^{k}(t-\bar{t}_{1k})t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)}{\frac{1}{NT^3}N\sum_{t=1}^{k}(t-\bar{t}_{1k})^2}+\frac{\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T^{5/2}}\sum_{t=1}^{k}(t-\bar{t}_{1k})v_{it}}{\frac{1}{NT^3}N\sum_{t=1}^{k}(t-\bar{t}_{1k})^2} \\ =\; & 12\left(\frac{k}{T}\right)^{-3}\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T^{5/2}}\sum_{t=1}^{k}(t-\bar{t}_{1k})t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)+\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T^{5/2}}\sum_{t=1}^{k}(t-\bar{t}_{1k})v_{it}\right]+o_p(1) \end{aligned}$$

Similarly

$$\begin{aligned} & \sqrt{NT}\left(\widehat{\beta}_T^{(T)}-\beta\right) \\ =\; & 12\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T^{5/2}}\sum_{t=1}^{T}(t-\bar{t})t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)+\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T^{5/2}}\sum_{t=1}^{T}(t-\bar{t})v_{it}+o_p(1) \end{aligned}$$

which implies that

$$\begin{aligned}
&\sqrt{NT}\left(\widehat{\beta}_k^{(T)}-\widehat{\beta}_T^{(T)}\right)\\
=\ &\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[12\left(\frac{k}{T}\right)^{-3}\frac{1}{T^{5/2}}\sum_{t=1}^{k}(t-\bar{t}_{1k})v_{it}-\frac{12}{T^{5/2}}\sum_{t=1}^{T}(t-\bar{t})v_{it}\right]\\
&+\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[12\left(\frac{k}{T}\right)^{-3}\frac{1}{T^{5/2}}\sum_{t=1}^{k}(t-\bar{t}_{1k})t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)-\frac{12}{T^{5/2}}\sum_{t=1}^{T}(t-\bar{t})t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)\right]+o_p(1)
\end{aligned}$$

From Theorem 3 we know that

$$\frac{1}{6\sigma_\varepsilon}\left[\frac{1}{T^{5/2}}\sum_{t=1}^{k}(t-\bar{t}_{1k})v_{it}-\left(\frac{k}{T}\right)^3\frac{1}{T^{5/2}}\sum_{t=1}^{T}(t-\bar{t})v_{it}\right]\xrightarrow{d}H_0(r)$$

and

$$\begin{aligned}
&\frac{1}{T^{5/2}}\sum_{t=1}^{k}(t-\bar{t}_{1k})t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)-\left(\frac{k}{T}\right)^3\frac{1}{T^{5/2}}\sum_{t=1}^{T}(t-\bar{t})t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)\\
\to\ &\left(\int_0^r s^2g(s)ds-\frac{1}{2}r\int_0^r sg(s)ds\right)-r^3\left(\int_0^1 s^2g(s)ds-\frac{1}{2}\int_0^1 sg(s)ds\right)
\end{aligned}$$

uniformly in r. Then

$$\begin{aligned}
&\sqrt{NT}\frac{1}{6\sigma_\varepsilon}\left(\frac{k}{T}\right)^3\left(\widehat{\beta}_k^{(T)}-\widehat{\beta}_T^{(T)}\right)\\
&\xrightarrow{d}H_0(r)+\frac{2}{\sigma_\varepsilon}\left[\left(\int_0^r s^2g(s)ds-\frac{1}{2}r\int_0^r sg(s)ds\right)-r^3\left(\int_0^1 s^2g(s)ds-\frac{1}{2}\int_0^1 sg(s)ds\right)\right]
\end{aligned}$$

uniformly in r for all N proving (b).

To prove (c):

$$\begin{aligned}
&\sqrt{NT}\left(\widehat{\beta}_k^{(T)}-\beta\right)\\
=\ &\frac{\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T^{5/2}}\sum_{t=1}^{k}(t-\bar{t}_{1k})t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)}{\frac{1}{NT^3}N\sum_{t=1}^{k}(t-\bar{t}_{1k})^2}+\frac{\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T^{5/2}}\sum_{t=1}^{k}(t-\bar{t}_{1k})v_{it}}{\frac{1}{NT^3}N\sum_{t=1}^{k}(t-\bar{t}_{1k})^2}\\
=\ &12\left(\frac{k}{T}\right)^{-3}\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T^{5/2}}\sum_{t=1}^{k}(t-\bar{t}_{1k})t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)+\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T^{5/2}}\sum_{t=1}^{k}(t-\bar{t}_{1k})v_{it}\right]+o_p(1)
\end{aligned}$$

Similarly

$$\begin{aligned}
&\sqrt{NT}\left(\widehat{\beta}_T^{(T)}-\beta\right)\\
=\ &12\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T^{5/2}}\sum_{t=1}^{T}(t-\bar{t})t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)+\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T^{5/2}}\sum_{t=1}^{T}(t-\bar{t})v_{it}+o_p(1)
\end{aligned}$$

which implies that

$$\begin{aligned}
& \sqrt{NT}\left(\widehat{\beta}_k^{(T)}-\widehat{\beta}_T^{(T)}\right) \\
= & \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[12\left(\frac{k}{T}\right)^{-3}\frac{1}{T^{5/2}}\sum_{t=1}^{k}(t-\bar{t}_{1k})v_{it}-\frac{12}{T^{5/2}}\sum_{t=1}^{T}(t-\bar{t})v_{it}\right] \\
& +\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[12\left(\frac{k}{T}\right)^{-3}\frac{1}{T^{5/2}}\sum_{t=1}^{k}(t-\bar{t}_{1k})t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)-\frac{12}{T^{5/2}}\sum_{t=1}^{T}(t-\bar{t})t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)\right]+o_p\left(1\right)
\end{aligned}$$

From Theorem 5 we know that

$$\frac{1}{6\sigma_\varepsilon}\left[\frac{1}{T^{5/2}}\sum_{t=1}^{k}(t-\bar{t}_{1k})v_{it}-\left(\frac{k}{T}\right)^3\frac{1}{T^{5/2}}\sum_{t=1}^{T}(t-\bar{t})v_{it}\right]\xrightarrow{d}H_c(r)-r^3H_c(1)$$

and

$$\begin{aligned}
& \frac{1}{T^{5/2}}\sum_{t=1}^{k}(t-\bar{t}_{1k})t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)-\left(\frac{k}{T}\right)^3\frac{1}{T^{5/2}}\sum_{t=1}^{T}(t-\bar{t})t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right) \\
\to & \left(\int_0^r s^2g(s)ds-\frac{1}{2}r\int_0^r sg(s)ds\right)-r^3\left(\int_0^1 s^2g(s)ds-\frac{1}{2}\int_0^1 sg(s)ds\right)
\end{aligned}$$

uniformly in r. Then

$$\begin{aligned}
& \sqrt{NT}\frac{1}{6\sigma_\varepsilon}\left(\frac{k}{T}\right)^3\left(\widehat{\beta}_k^{(T)}-\widehat{\beta}_T^{(T)}\right) \\
& \xrightarrow{d}H_c(r)-r^3H_c(1)+\frac{2}{\sigma_\varepsilon}\left[\left(\int_0^r s^2g(s)ds-\frac{1}{2}r\int_0^r sg(s)ds\right)-r^3\left(\int_0^1 s^2g(s)ds-\frac{1}{2}\int_0^1 sg(s)ds\right)\right]
\end{aligned}$$

uniformly in r for all N proving (c). ■

H Proof of Theorem 8

Proof. Under the local alternative we have

$$\begin{aligned}
& \sqrt{NT^3}\left(\widehat{\beta}_{1k}^{(T)}-\beta\right) \\
= & 12\left(\frac{k}{T}\right)^{-3}\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T^{3/2}}\sum_{t=1}^{k}(t-\bar{t}_{1k})t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)+\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T^{3/2}}\sum_{t=1}^{k}(t-\bar{t}_{1k})v_{it}\right]+o_p(1) \\
& \xrightarrow{d}6r^{-3}\left[\sigma_0G(r)+O_p\left(T\right)\right]
\end{aligned}$$

and

$$\sqrt{NT^3}\left(\widehat{\beta}_{2k}^{(T)}-\beta\right)\xrightarrow{d}6\left(1-r\right)^{-3}\left[\sigma_0\left[G(1)-G(r)-rW(1)+W(r)\right]+O_p\left(T\right)\right],$$

uniformly in r where

$$G(r) = rW(r) - 2\int_0^r W(r)dr.$$

Part (a) holds using the proof of Theorem 2 with $G(r)$ and $G(1) - G(r) - rW(1) + W(r)$ replaced by

$$G(r) + O_p(T)$$

and

$$G(1) - G(r) - rW(1) + W(r) + O_p(T),$$

respectively. Then under the alternative

$$\begin{aligned}
&\mathrm{W}_1^{(T)}(k) \\
&\xrightarrow{d} \left\{ \frac{(G(r) + O_p(T))(1-r)^3 - r^3(G(1) - G(r) - rW(1) + W(r) + O_p(T))}{\left[r^3(1-r)^3\left[(1-r)^3 + r^3\right]\right]^{\frac{1}{2}}} \right\}^2 \\
= &\left\{ \frac{\left(\left[(1-r)^3 + r^3\right]G(r) - r^3\left[G(1) - rW(1) + W(r)\right]\right)}{\left[r^3(1-r)^3\left[(1-r)^3 + r^3\right]\right]^{\frac{1}{2}}} + O_p(T) \right\}^2
\end{aligned}$$

uniformly in r.

Part (b) holds using the proof of Theorem 4 with $H(r)$ and $H(1) - H(r) - r\int_0^1 W(s)ds + \int_0^r W(s)ds$ replaced by

$$H(r) + 2\frac{h(r)}{\sigma_\varepsilon}$$

and

$$H(1) - H(r) - r\int_0^1 W(s)ds + \int_0^r W(s)ds + \frac{2}{\sigma_\varepsilon}\left(h(1) - h(r) - \frac{1}{2}r\int_0^1 sg(s)ds + \frac{1}{2}\int_0^r sg(s)ds\right),$$

respectively. Then under the alternative

$$
\begin{aligned}
&\mathrm{W}_2^{(T)}(k) \\
&\xrightarrow{d} \left\{ \frac{\left(H(r)+2\frac{h(r)}{\sigma_\varepsilon}\right)(1-r)^3 - r^3 \left(\begin{array}{c} H(1)-H(r)-r\int_0^1 W(s)ds + \int_0^r W(s)ds \\ +\frac{2}{\sigma_\varepsilon}\left(h(1)-h(r)-\frac{1}{2}r\int_0^1 sg(s)ds+\frac{1}{2}\int_0^r sg(s)ds\right) \end{array} \right)}{\left[r^3(1-r)^3\left[(1-r)^3+r^3\right]\right]^{\frac{1}{2}}} \right\}^2 \\
&= \left\{ \frac{\left(\begin{array}{c} \left[(1-r)^3+r^3\right]H(r) \\ -r^3 \left[\begin{array}{c} H(1)-r\int_0^1 W(s)ds \\ +\int_0^r W(s)ds \end{array} \right] \end{array} \right)}{\left[r^3(1-r)^3\left[(1-r)^3+r^3\right]\right]^{\frac{1}{2}}} + \frac{2}{\sigma_\varepsilon} \frac{\left(\begin{array}{c} \left[(1-r)^3+r^3\right]h(r) \\ -r^3 \left[\begin{array}{c} h(1)-\frac{1}{2}r\int_0^1 sg(s)ds \\ +\frac{1}{2}\int_0^r sg(s)ds \end{array} \right] \end{array} \right)}{\left[r^3(1-r)^3\left[(1-r)^3+r^3\right]\right]^{\frac{1}{2}}} \right\}^2
\end{aligned}
$$

Part (c) holds using the proof of Theorem 6 with $H_c(r)$ and $H_c(1)-H_c(r)-r\int_0^1\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds+\int_0^r\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds$ replaced by

$$H_c(r)+2\frac{h(r)}{\sigma_\varepsilon}$$

and

$$
\begin{aligned}
&H_c(1)-H_c(r)-r\int_0^1\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds+\int_0^r\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds \\
&+\frac{2}{\sigma_\varepsilon}\left(h(1)-h(r)-\frac{1}{2}r\int_0^1 sg(s)ds+\frac{1}{2}\int_0^r sg(s)ds\right),
\end{aligned}
$$

respectively. Then under the alternative

$$
\begin{aligned}
&\mathrm{W}_c^{(T)}(k) \\
&\xrightarrow{d} \left\{ \frac{\left(H_c(r)+2\frac{h(r)}{\sigma_\varepsilon}\right)(1-r)^3 - r^3 \begin{pmatrix} H_c(1)-H_c(r)-r\int_0^1 \left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds \\ +\int_0^r \left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds \\ +\frac{2}{\sigma_\varepsilon}\begin{pmatrix} h(1)-h(r)-\frac{1}{2}r\int_0^1 sg(s)ds+ \\ \frac{1}{2}\int_0^r sg(s)ds \end{pmatrix}\end{pmatrix}}{\left[r^3(1-r)^3\left[(1-r)^3+r^3\right]\right]^{\frac{1}{2}}} \right\}^2 \\
&= \left\{ \frac{\begin{pmatrix} (1-r)^3 H_c(r) \\ -r^3 \begin{bmatrix} H_c(1)-H_c(r) \\ -r\int_0^1 \left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds \\ +\int_0^r \left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds \end{bmatrix}\end{pmatrix}}{\left[r^3(1-r)^3\left[(1-r)^3+r^3\right]\right]^{\frac{1}{2}}} + \frac{2}{\sigma_\varepsilon}\frac{\begin{pmatrix} (1-r)^3 h(r) \\ -r^3 \begin{bmatrix} h(1)-h(r) \\ -\frac{1}{2}r\int_0^1 sg(s)ds \\ +\frac{1}{2}\int_0^r sg(s)ds \end{bmatrix}\end{pmatrix}}{\left[r^3(1-r)^3\left[(1-r)^3+r^3\right]\right]^{\frac{1}{2}}} \right\}^2
\end{aligned}
$$

■

Lemma 1 *When* $|\rho|<1$, *we have*

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T^{5/2}}\sum_{t=1}^{k}t^2 v_{it} \xrightarrow{d} r^2 W(r) - r\int_0^r W(s)ds - \int_0^r sW(s)ds.$$

Proof. Note that

$$
\begin{aligned}
\frac{1}{T^{5/2}}\sum_{t=1}^{k}t^2 v_{it} &= \frac{1}{T^{5/2}}\sum_{t=1}^{k}k^2 v_{it} - \frac{1}{T^{5/2}}\sum_{t=1}^{k-1}\left(k^2-t^2\right)v_{it} \\
&= \frac{1}{T^{5/2}}\sum_{t=1}^{k}k^2 v_{it} - \frac{1}{T^{5/2}}\left(k\sum_{t=1}^{k-1}(k-t)\,v_{it} + \sum_{t=1}^{k-1}(k-t)\,t v_{it}\right) \\
&= \frac{1}{T^{5/2}}\sum_{t=1}^{k}k^2 v_{it} - \frac{1}{T^{5/2}}\left(k\sum_{j=2}^{k}\sum_{t=1}^{j-1}v_{it} + \sum_{j=2}^{k}\sum_{t=1}^{j-1}t v_{it}\right) \\
&\xrightarrow{d} r^2 W(r) - r\int_0^r W(s)ds - \int_0^r sW(s)ds.
\end{aligned}
$$

For a fixed N as $T \to \infty$ we have

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{1}{T^{5/2}} \sum_{t=1}^{k} t^2 v_{it} \xrightarrow{d} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left(r^2 W(r) - r \int_0^r W(s)ds - \int_0^r sW(s)ds \right).$$

Hence

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left(r^2 W(r) - r \int_0^r W(s)ds - \int_0^r sW(s)ds \right) \xrightarrow{d} r^2 W(r) - r \int_0^r W(s)ds - \int_0^r sW(s)ds$$

as $N \to \infty$. ■

I Proof of Theorem 9

Proof. We can now prove Theorem 9 using the lemma. First we note that

$$\left(\underline{\widehat{\beta}}_{(1k)} - \underline{\widehat{\beta}}_{(2k)} \right) = \left(\underline{\widehat{\beta}}_{(1k)} - \underline{\beta} \right) - \left(\underline{\widehat{\beta}}_{(2k)} - \underline{\beta} \right)$$

Define

$$F(r) = 2r^2 W(r) - 3r \int_0^r W(s)ds - 3 \int_0^r sW(s)ds$$

and define the following weight matrices

$$\omega_1 = \begin{bmatrix} \frac{1}{T^{1/2}} & 0 \\ 0 & \frac{1}{T^{3/2}} \end{bmatrix}$$

$$\omega_2 = \begin{bmatrix} \frac{1}{T^{3/2}} & 0 \\ 0 & \frac{1}{T^{5/2}} \end{bmatrix}.$$

and

$$\omega_{2*} = \begin{bmatrix} \frac{1}{T^{5/2}} & 0 \\ 0 & \frac{1}{T^{7/2}} \end{bmatrix}$$

Remark 4 *Note that for a panel polynomial regression of order p, the weight matrices will be as follows*

$$\omega_{1p} = \begin{bmatrix} \frac{1}{T^{1/2}} & 0 & \cdots & 0 \\ 0 & \frac{1}{T^{3/2}} & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{T^{(2p-1)/2}} \end{bmatrix}.$$

$$\omega_{2p} = \begin{bmatrix} \frac{1}{T^{3/2}} & 0 & \cdots & 0 \\ 0 & \frac{1}{T^{5/2}} & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{T^{(2p+1)/2}} \end{bmatrix}.$$

and

$$\omega_{2p*} = \begin{bmatrix} \frac{1}{T^{5/2}} & 0 & \cdots & 0 \\ 0 & \frac{1}{T^{7/2}} & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{T^{(2p+3)/2}} \end{bmatrix}.$$

Then under H_0 and $|\rho| < 1$ we have

$$\begin{aligned}
& \sqrt{N}\omega_2^{-1}\left(\underline{\widehat{\beta}}_{(1k)} - \underline{\widehat{\beta}}_{(2k)}\right) \\
= \ & \sqrt{N}\omega_2^{-1}\left(\underline{\widehat{\beta}}_{(1k)} - \underline{\beta}\right) - \sqrt{N}\omega_2^{-1}\left(\underline{\widehat{\beta}}_{(2k)} - \underline{\beta}\right) \\
= \ & \left[\frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{k}\omega_2\left(\underline{t} - \underline{\overline{t}}_{(1k)}\right)'\left(\underline{t} - \underline{\overline{t}}_{(1k)}\right)\omega_2\right]^{-1}\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\sum_{t=1}^{k}\omega_2\left(\underline{t} - \underline{\overline{t}}_{(1k)}\right)' v_{it}\right] \\
& - \left[\frac{1}{N}\sum_{i=1}^{N}\sum_{t=k+1}^{T}\omega_2\left(\underline{t} - \underline{\overline{t}}_{(2k)}\right)'\left(\underline{t} - \underline{\overline{t}}_{(2k)}\right)\omega_2\right]^{-1}\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\sum_{t=k+1}^{T}\omega_2\left(\underline{t} - \underline{\overline{t}}_{(2k)}\right)' v_{it}\right] \\
& \xrightarrow{d} \begin{bmatrix} \frac{1}{12}r^3 & \frac{1}{12}r^4 \\ \frac{1}{12}r^4 & \frac{4}{45}r^5 \end{bmatrix}^{-1}\begin{bmatrix} \frac{1}{2}\sigma_0 G(r) \\ \frac{1}{3}\sigma_0 F(r) \end{bmatrix} \\
& - \begin{bmatrix} \frac{(1-r)^3}{12} & \frac{(1-r)^3(1+r)}{12} \\ \frac{(1-r)^3(1+r)}{12} & \frac{\left(4+7r+r^2\right)(1-r)^3}{45} \end{bmatrix}^{-1}\begin{bmatrix} \frac{1}{2}\sigma_0\left(G(1) - G(r) + W(r) - rW(1)\right) \\ \frac{1}{3}\sigma_0\left[F(1) - F(r) + (1+r)W(r) - r(1+r)W(1)\right] \end{bmatrix} \\
= \ & \frac{\sigma_0}{r^4}\begin{bmatrix} \frac{-12A}{C} \\ \frac{30B}{rC} \end{bmatrix}.
\end{aligned}$$

where

$$\begin{gathered}
A = G(r)\left[-2r\left(45r^5 - 117r^4 + 108r^3 - 8r^2 - 20r + 4\right)\right] \\
+F(r)\left[60r^5 - 150r^4 + 130r^3 - 10r^2 - 25r + 5\right] + W(r)\left[-r^4\left(3r^2 - 4r - 3\right)\right] \\
+G(1)\left[2r^4\left(r^2 + 7r + 4\right)\right] + F(1)\left[-5r^4(r+1)\right] + W(1)\left[r^5\left(3r^2 - 4r - 3\right)\right],
\end{gathered}$$

$$\begin{gathered}
B = G(r)\left[-3r\left(12r^5 - 30r^4 + 1 - 5r - 2r^2 + 26r^3\right)\right] \\
+F(r)\left[24r^5 - 62r^4 + 52r^3 - 4r^2 - 10r + 2\right] \\
+W(r)\left[r^5(r+1)\right] + G(1)\left[3r^5(r+1)\right] + F(1)\left[-2r^5\right] + W(1)\left[-r^6(r+1)\right],
\end{gathered}$$

and

$$C = 11r^5 - 31r^4 + 26r^3 - 2r^2 - 5r + 1.$$

Now,

$$N\omega_2^{-1}\omega_2^{-1}\left(Var\left(b_a\right)\right)$$
$$= \begin{bmatrix} \left[\frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{k}\omega_2\left(\underline{t}-\underline{\overline{t}}_{(1k)}\right)'\left(\underline{t}-\underline{\overline{t}}_{(1k)}\right)\omega_2\right]^{-1} & 0_{2\times 2} \\ 0_{2\times 2} & \left[\frac{1}{N}\sum_{i=1}^{N}\sum_{t=k+1}^{T}\omega_2\left(\underline{t}-\underline{\overline{t}}_{(2k)}\right)'\left(\underline{t}-\underline{\overline{t}}_{(2k)}\right)\omega_2\right]^{-1} \end{bmatrix}$$
$$\to \begin{bmatrix} \begin{bmatrix} \frac{1}{12}r^3 & \frac{1}{12}r^4 \\ \frac{1}{12}r^4 & \frac{4}{45}r^5 \end{bmatrix}^{-1} & 0_{2\times 2} \\ 0_{2\times 2} & \begin{bmatrix} \frac{(1-r)^3}{12} & \frac{(1-r)^3(1+r)}{12} \\ \frac{(1-r)^3(1+r)}{12} & \frac{\left(4+7r+r^2\right)(1-r)^3}{45} \end{bmatrix}^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} \begin{bmatrix} \frac{192}{r^3} & -\frac{180}{r^4} \\ -\frac{180}{r^4} & \frac{180}{r^5} \end{bmatrix} & 0_{2\times 2} \\ 0_{2\times 2} & \begin{bmatrix} 48\frac{4+7r+r^2}{11r^5-31r^4+26r^3-2r^2-5r+1} & -180\frac{1+r}{11r^5-31r^4+26r^3-2r^2-5r+1} \\ -180\frac{1+r}{11r^5-31r^4+26r^3-2r^2-5r+1} & \frac{180}{11r^5-31r^4+26r^3-2r^2-5r+1} \end{bmatrix} \end{bmatrix}$$

Then

$$N\omega_2^{-1}\omega_2^{-1}\left[R(Var(b_a))R'\ \right]$$
$$\to \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{192}{r^3} & -\frac{180}{r^4} & 0 & 0 \\ -\frac{180}{r^4} & \frac{180}{r^5} & 0 & 0 \\ 0 & 0 & \frac{48\left(4+7r+r^2\right)}{11r^5-31r^4+26r^3-2r^2-5r+1} & \frac{-180(1+r)}{11r^5-31r^4+26r^3-2r^2-5r+1} \\ 0 & 0 & \frac{-180(1+r)}{11r^5-31r^4+26r^3-2r^2-5r+1} & \frac{180}{11r^5-31r^4+26r^3-2r^2-5r+1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 48\frac{45r^5-117r^4+108r^3-8r^2-20r+4}{r^3(11r^2+2r-1)(r-1)^3} & -180\frac{12r^5-30r^4+26r^3-2r^2-5r+1}{r^4(11r^2+2r-1)(r-1)^3} \\ -180\frac{12r^5-30r^4+26r^3-2r^2-5r+1}{r^4(11r^2+2r-1)(r-1)^3} & 180\frac{12r^5-31r^4+26r^3-2r^2-5r+1}{r^5(11r^2+2r-1)(r-1)^3} \end{bmatrix}.$$

This gives

$$\frac{1}{N}\omega_2\omega_2\left[R(Var(b_a))R'\ \right]^{-1} \to \frac{r^3}{3a}\begin{bmatrix} -\frac{1}{4}c & -\frac{1}{4}rb \\ -\frac{1}{4}rb & -\frac{1}{15}r^2 d \end{bmatrix}$$

where

$$a = 36r^7 - 144r^6 + 228r^5 - 160r^4 + 32r^3 + 16r^2 - 8r + 1,$$

$$b = \left(12r^5 - 30r^4 + 26r^3 - 2r^2 - 5r + 1\right)\left(r^3 - 3r^2 + 3r - 1\right),$$

$$c = \left(12r^5 - 31r^4 + 26r^3 - 2r^2 - 5r + 1\right)\left(r^3 - 3r^2 + 3r - 1\right),$$

and

$$d = \left(45r^5 - 117r^4 + 108r^3 - 8r^2 - 20r + 4\right)\left(r^3 - 3r^2 + 3r - 1\right).$$

Thus we have

$$\begin{aligned} W_{1p}(k) &\xrightarrow{d} \frac{1}{3r^5 a}\begin{bmatrix} \frac{-12A}{C} & \frac{30B}{rC} \end{bmatrix}\begin{bmatrix} -\frac{1}{4}c & -\frac{1}{4}rb \\ -\frac{1}{4}rb & -\frac{1}{15}r^2 d \end{bmatrix}\begin{bmatrix} \frac{-12A}{C} \\ \frac{30B}{rC} \end{bmatrix} \\ &= 4\frac{\left(-3A^2c + 15ABb - 5B^2d\right)}{r^5 a C^2} \\ &= P_1(r), \end{aligned}$$

where

$$P_1(r) = 4\frac{\left(-3A^2c + 15ABb - 5B^2d\right)}{r^5 a C^2}.$$

Next we consider the case when $\rho = 1$. Define

$$J(r) = 3\int_0^r s^2\left[W(s) + \widetilde{W}(\kappa)\right] ds - r^2\int_0^r \left[W(s) + \widetilde{W}(\kappa)\right] ds.$$

Then under H_0 we have

$$\begin{aligned}
& \sqrt{N}\omega_1^{-1}\left(\underline{\widehat{\beta}}_{(1k)}-\underline{\widehat{\beta}}_{(2k)}\right) \\
= & \sqrt{N}\omega_1^{-1}\left(\underline{\widehat{\beta}}_{(1k)}-\underline{\beta}\right)-\sqrt{N}\omega_1^{-1}\left(\underline{\widehat{\beta}}_{(2k)}-\underline{\beta}\right) \\
= & \left[\frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{k}\omega_2\left(\underline{t}-\underline{\overline{t}}_{(1k)}\right)'\left(\underline{t}-\underline{\overline{t}}_{(1k)}\right)\omega_2\right]^{-1}\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\sum_{t=1}^{k}\omega_{2*}\left(\underline{t}-\underline{\overline{t}}_{(1k)}\right)'v_{it}\right] \\
& -\left[\frac{1}{N}\sum_{i=1}^{N}\sum_{t=k+1}^{T}\omega_2\left(\underline{t}-\underline{\overline{t}}_{(2k)}\right)'\left(\underline{t}-\underline{\overline{t}}_{(2k)}\right)\omega_2\right]^{-1}\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\sum_{t=k+1}^{T}\omega_{2*}\left(\underline{t}-\underline{\overline{t}}_{(2k)}\right)'v_{it}\right] \\
& \xrightarrow{d}\left[\begin{array}{cc}\frac{1}{12}r^3 & \frac{1}{12}r^4 \\ \frac{1}{12}r^4 & \frac{4}{45}r^5\end{array}\right]^{-1}\left[\begin{array}{c}\frac{1}{2}\sigma_\varepsilon H(r) \\ \frac{1}{3}\sigma_\varepsilon J(r)\end{array}\right] \\
& -\left[\begin{array}{cc}\frac{(1-r)^3}{12} & \frac{(1-r)^3(1+r)}{12} \\ \frac{(1-r)^3(1+r)}{12} & \frac{\left(4+7r+r^2\right)(1-r)^3}{45}\end{array}\right]^{-1}\left[\begin{array}{c}\frac{1}{2}\sigma_\varepsilon\left(\begin{array}{c}H(1)-H(r)- \\ r\int_0^1 W(s)ds+\int_0^r W(s)ds\end{array}\right) \\ \frac{1}{3}\sigma_\varepsilon\left(\begin{array}{c}J(1)-J(r)- \\ r\left(1+r\right)\int_0^1 W(s)ds+ \\ \left(1+r\right)\int_0^r W(s)ds\end{array}\right)\end{array}\right] \\
= & \frac{\sigma_\varepsilon}{r^4}\left[\begin{array}{c}\frac{-12D}{C} \\ \frac{30E}{rC}\end{array}\right]
\end{aligned}$$

where

$$\begin{aligned}
D = & H(r)\left[-2r\left(4-20r+45r^5-117r^4+108r^3-8r^2\right)\right]+J(r)\left[60r^5-150r^4+130r^3-10r^2-25r+5\right]+ \\
& H(1)\left[2r^4\left(4+7r+r^2\right)\right]+J(1)\left[-5r^4\left(r+1\right)\right]+ \\
& \left(r^5\left(-3-4r+3r^2\right)\right)\int_0^1 W(s)ds+\left(-r^4\left(-3-4r+3r^2\right)\right)\int_0^r W(s)ds
\end{aligned}$$

and

$$\begin{aligned}
E = & H(r)\left[-3r\left(1-5r+12r^5-30r^4+26r^3-2r^2\right)\right]+J(r)\left[24r^5-62r^4+52r^3-4r^2-10r+2\right]+ \\
& H(1)\left[3r^5\left(r+1\right)\right]+J(1)\left[-2r^5\right]+\left(-r^6\left(r+1\right)\right)\int_0^1 W(s)ds+\left(r^5\left(r+1\right)\right)\int_0^r W(s)ds.
\end{aligned}$$

Then

$$\begin{aligned}
& \frac{1}{T^2}W_2 p(k)\xrightarrow{d}\frac{1}{3r^5a}\left[\begin{array}{cc}\frac{-12D}{C} & \frac{30E}{rC}\end{array}\right]\left[\begin{array}{cc}-\frac{1}{4}c & -\frac{1}{4}rb \\ -\frac{1}{4}rb & -\frac{1}{15}r^2d\end{array}\right]\left[\begin{array}{c}\frac{-12D}{C} \\ \frac{30E}{rC}\end{array}\right] \\
= & 4\frac{\left(-3D^2c+15DEb-5E^2d\right)}{r^5aC^2} \\
= & P_2\left(r\right)
\end{aligned}$$

where

$$P_2(r) = 4\frac{\left(-3D^2c + 15DEb - 5E^2d\right)}{r^5aC^2}.$$

Finally we consider the case when $\rho = 1 + \frac{c}{T}$. Define

$$J_c(r) = 3\int_0^r s^2\left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa)\right]ds - r^2\int_0^r\left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa)\right]ds.$$

Then under H_0 we have

$$\begin{aligned}
& \sqrt{N}\omega_1^{-1}\left(\underline{\widehat{\beta}}_{(1k)} - \underline{\widehat{\beta}}_{(2k)}\right) \\
= & \sqrt{N}\omega_1^{-1}\left(\underline{\widehat{\beta}}_{(1k)} - \underline{\beta}\right) - \sqrt{N}\omega_1^{-1}\left(\underline{\widehat{\beta}}_{(2k)} - \underline{\beta}\right) \\
= & \left[\frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{k}\omega_2\left(\underline{t} - \underline{\overline{t}}_{(1k)}\right)\left(\underline{t} - \underline{\overline{t}}_{(1k)}\right)\omega_2\right]^{-1}\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\sum_{t=1}^{k}\omega_{2*}\left(\underline{t} - \underline{\overline{t}}_{(1k)}\right)' v_{it}\right] - \\
& \left[\frac{1}{N}\sum_{i=1}^{N}\sum_{t=k+1}^{T}\omega_2\left(\underline{t} - \underline{\overline{t}}_{(2k)}\right)'\left(\underline{t} - \underline{\overline{t}}_{(2k)}\right)\omega_2\right]^{-1}\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\sum_{t=k+1}^{T}\omega_{2*}\left(\underline{t} - \underline{\overline{t}}_{(2k)}\right)' v_{it}\right] \\
& \xrightarrow{d} \begin{bmatrix} \frac{1}{12}r^3 & \frac{1}{12}r^4 \\ \frac{1}{12}r^4 & \frac{4}{45}r^5 \end{bmatrix}^{-1}\begin{bmatrix} \frac{1}{2}\sigma_\varepsilon H_c(r) \\ \frac{1}{3}\sigma_\varepsilon J_c(r) \end{bmatrix} - \begin{bmatrix} \frac{(1-r)^3}{12} & \frac{(1-r)^3(1+r)}{12} \\ \frac{(1-r)^3(1+r)}{12} & \frac{\left(4+7r+r^2\right)(1-r)^3}{45} \end{bmatrix}^{-1} \\
& \begin{bmatrix} \frac{1}{2}\sigma_\varepsilon\begin{pmatrix} H_c(1) - H_c(r) - \\ r\int_0^1\left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa)\right]ds + \\ \int_0^r\left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa)\right]ds \end{pmatrix} \\ \frac{1}{3}\sigma_\varepsilon\begin{pmatrix} J_c(1) - J_c(r) - \\ r\left(1+r\right)\int_0^1\left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa)\right]ds + \\ \left(1+r\right)\int_0^r\left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa)\right]ds \end{pmatrix} \end{bmatrix} \\
= & \frac{\sigma_\varepsilon}{r^4}\begin{pmatrix} \frac{-12F}{C} \\ \frac{30G}{rC} \end{pmatrix}
\end{aligned}$$

where

$$\begin{aligned}
F = {} & H_c(r)\left[-2r\left(4 - 20r + 45r^5 - 117r^4 + 108r^3 - 8r^2\right)\right] + J_c(r)\left[60r^5 - 150r^4 + 130r^3 - 10r^2 - 25r + 5\right] + \\
& H_c(1)\left[2r^4\left(4 + 7r + r^2\right)\right] + J_c(1)\left[-5r^4\left(r+1\right)\right] + \left(r^5\left(-3 - 4r + 3r^2\right)\right)\int_0^1\left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa)\right]ds + \\
& \left(-r^4\left(-3 - 4r + 3r^2\right)\right)\int_0^r\left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa)\right]ds
\end{aligned}$$

and

$$G = H_c(r)\left[-3r\left(1-5r+12r^5-30r^4+26r^3-2r^2\right)\right]+J_c(r)\left[24r^5-62r^4+52r^3-4r^2-10r+2\right]+$$
$$H_c(1)\left[3r^5\left(r+1\right)\right]+J_c(1)\left[2r^5\right]+\left(-r^6\left(r+1\right)\right)\int_0^1\left[W_c(s)+e^{sc}\widetilde{W_c}(\kappa)\right]ds+$$
$$\left(r^5\left(r+1\right)\right)\int_0^r\left[W_c(s)+e^{sc}\widetilde{W_c}(\kappa)\right]ds.$$

Then

$$\begin{aligned}
\frac{1}{T^2}W_{cp}(k) &\xrightarrow{d} \frac{1}{3r^5a}\begin{bmatrix} \frac{-12F}{C} & \frac{30G}{rC}\end{bmatrix}\begin{bmatrix} -\frac{1}{4}c & -\frac{1}{4}rb \\ -\frac{1}{4}rb & -\frac{1}{15}r^2d\end{bmatrix}\begin{bmatrix} \frac{-12F}{C} \\ \frac{30G}{rC}\end{bmatrix} \\
&= 4\frac{\left(-3F^2c+-15FGb-5G^2d\right)}{r^5aC^2} \\
&= 4P_c\left(r\right),
\end{aligned}$$

where

$$P_c\left(r\right)=4\frac{\left(-3F^2c+-15FGb-5G^2d\right)}{r^5aC^2}.$$

■

J Proof of Theorem 10

Proof. Under the local alternative we have

$$\begin{aligned}
&\sqrt{N}\omega_2^{-1}\left(\widehat{\underline{\beta}}_{(1k)}^{(T)}-\underline{\beta}\right) \\
&= \left[\frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{k}\omega_2\left(\underline{t}-\underline{\overline{t}}_{(1k)}\right)'\left(\underline{t}-\underline{\overline{t}}_{(1k)}\right)\omega_2\right]^{-1} \\
&\quad\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\sum_{t=1}^{k}\omega_2\left(\underline{t}-\underline{\overline{t}}_{(1k)}\right)'\left(t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)+t^2\frac{1}{T^{3/2}}g\left(\frac{t}{T}\right)+v_{it}\right)\right] \\
&\xrightarrow{d} \begin{bmatrix} \frac{192}{r^3} & -\frac{180}{r^4} \\ -\frac{180}{r^4} & \frac{180}{r^5}\end{bmatrix}\begin{bmatrix} O_p\left(T\right) \\ O_p\left(T\right)\end{bmatrix}+\begin{bmatrix} \frac{192}{r^3} & -\frac{180}{r^4} \\ -\frac{180}{r^4} & \frac{180}{r^5}\end{bmatrix}\begin{bmatrix} O_p\left(T\right) \\ O_p\left(T\right)\end{bmatrix} \\
&\qquad\begin{bmatrix} \frac{192}{r^3} & -\frac{180}{r^4} \\ -\frac{180}{r^4} & \frac{180}{r^5}\end{bmatrix}\begin{bmatrix} \frac{1}{2}\sigma_0G(r) \\ \frac{1}{3}\sigma_0F(r)\end{bmatrix} \\
&= \begin{bmatrix} 12\sigma_0\frac{8G(r)r-5F(r)}{r^4} \\ -30\sigma_0\frac{3G(r)r-2F(r)}{r^5}\end{bmatrix}+\begin{bmatrix} O_p\left(T\right) \\ O_p\left(T\right)\end{bmatrix}
\end{aligned}$$

We also have

$$
\begin{aligned}
&\sqrt{N}\omega_2^{-1}\left(\underline{\widehat{\beta}}_{(2k)}^{(T)}-\underline{\beta}\right)\\
=\;&\left[\frac{1}{N}\sum_{i=1}^{N}\sum_{t=k+1}^{T}\omega_2\left(\underline{t}-\underline{\overline{t}}_{(2k)}\right)'\left(\underline{t}-\underline{\overline{t}}_{(2k)}\right)\omega_2\right]^{-1}\\
&\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\sum_{t=k+1}^{T}\omega_2\left(\underline{t}-\underline{\overline{t}}_{(2k)}\right)'\left(t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)+t^2\frac{1}{T^{3/2}}g\left(\frac{t}{T}\right)+v_{it}\right)\right]\\
&\xrightarrow{d}\begin{bmatrix}\frac{48\left(4+7r+r^2\right)}{C} & \frac{-180(1+r)}{C}\\ \frac{-180(1+r)}{C} & \frac{180}{C}\end{bmatrix}\begin{bmatrix}O_p(T)\\ O_p(T)\end{bmatrix}+\begin{bmatrix}\frac{48\left(4+7r+r^2\right)}{C} & \frac{-180(1+r)}{C}\\ \frac{-180(1+r)}{C} & \frac{180}{C}\end{bmatrix}\begin{bmatrix}O_p(T)\\ O_p(T)\end{bmatrix}\\
&\begin{bmatrix}\frac{48\left(4+7r+r^2\right)}{C} & \frac{-180(1+r)}{C}\\ \frac{-180(1+r)}{C} & \frac{180}{C}\end{bmatrix}\begin{bmatrix}\frac{1}{2}\sigma_0\left(G(1)-G(r)+W(r)-rW(1)\right)\\ \frac{1}{3}\sigma_0\left(F(1)-F(r)+(1+r)W(r)-r(1+r)W(1)\right)\end{bmatrix}\\
=\;&\begin{bmatrix}\frac{12\sigma_0}{C}\begin{pmatrix}8G(1)-8G(r)+3W(r)-3rW(1)+14rG(1)-\\ 14G(r)r+4W(r)r-4r^2W(1)+2r^2G(1)-2r^2G(r)-\\ 3r^2W(r)+3r^3W(1)-5rF(1)+5rF(r)-5F(1)+5F(r)\end{pmatrix}\\ \frac{-30\sigma_0}{C}\begin{pmatrix}3rG(1)-3G(r)r+W(r)r-r^2W(1)+3G(1)-\\ 3G(r)+W(r)-rW(1)-2F(1)+2F(r)\end{pmatrix}\end{bmatrix}+\begin{bmatrix}O_p(T)\\ O_p(T)\end{bmatrix}
\end{aligned}
$$

Therefore, under the local alternative

$$W_{1p}^{(T)}(k)\xrightarrow{d}\left[P_1(r)+O_p(T)\right],$$

uniformly in r for all N proving (a).

Part (b) holds using the proof of Theorem 9 with

$$\begin{bmatrix}\frac{1}{2}\sigma_\varepsilon H(r)\\ \frac{1}{3}\sigma_\varepsilon J(r)\end{bmatrix}$$

and

$$\begin{bmatrix}\frac{1}{2}\sigma_\varepsilon\left(H(1)-H(r)-r\int_0^1 W(s)ds+\int_0^r W(s)ds\right)\\ \frac{1}{3}\sigma_\varepsilon\left(J(1)-J(r)-r(1+r)\int_0^1 W(s)ds+(1+r)\int_0^r W(s)ds\right)\end{bmatrix}$$

replaced by

$$\begin{bmatrix}\frac{1}{2}\sigma_\varepsilon H(r)\\ \frac{1}{3}\sigma_\varepsilon J(r)\end{bmatrix}+\begin{bmatrix}h_1(r)\\ h_3(r)\end{bmatrix}+\begin{bmatrix}h_2(r)\\ h_4(r)\end{bmatrix}$$

and

$$\begin{bmatrix} \frac{1}{2}\sigma_\varepsilon \left(H(1) - H(r) - r\int_0^1 W(s)ds + \int_0^r W(s)ds \right) \\ \frac{1}{3}\sigma_\varepsilon \left(J(1) - J(r) - r(1+r)\int_0^1 W(s)ds + (1+r)\int_0^r W(s)ds \right) \end{bmatrix} + \begin{bmatrix} e \\ g \end{bmatrix} + \begin{bmatrix} f \\ h \end{bmatrix}$$

respectively. Then under the local alternative

$$W_{2p}^{(T)}(k) \xrightarrow{d} \left[P_2(r) + \frac{4}{\sigma_\varepsilon^2} R_1(r) \right],$$

where

$$R_1(r) = \begin{bmatrix} \left[\left(\begin{bmatrix} h_1(r) \\ h_3(r) \end{bmatrix} + \begin{bmatrix} h_2(r) \\ h_4(r) \end{bmatrix} \right) - \left(\begin{bmatrix} e \\ g \end{bmatrix} + \begin{bmatrix} f \\ h \end{bmatrix} \right) \right]' \begin{bmatrix} -\frac{1}{12}\frac{r^3 c}{a} & -\frac{1}{12}\frac{r^4 b}{a} \\ -\frac{1}{12}\frac{r^4 b}{a} & -\frac{1}{45}\frac{r^5 d}{a} \end{bmatrix} \\ \left[\left(\begin{bmatrix} h_1(r) \\ h_3(r) \end{bmatrix} + \begin{bmatrix} h_2(r) \\ h_4(r) \end{bmatrix} \right) - \left(\begin{bmatrix} e \\ g \end{bmatrix} + \begin{bmatrix} f \\ h \end{bmatrix} \right) \right] \end{bmatrix},$$

$$\begin{bmatrix} h_1(r) \\ h_3(r) \end{bmatrix} = \begin{bmatrix} \int_0^r s^2 g(s)\,ds - \frac{1}{2}r\int_0^r sg(s)\,ds \\ \int_0^r s^3 g(s)\,ds - \frac{1}{3}r^2\int_0^r sg(s)\,ds \end{bmatrix},$$

$$\begin{bmatrix} h_2(r) \\ h_4(r) \end{bmatrix} = \begin{bmatrix} \int_0^r s^3 g(s)\,ds - \frac{1}{2}r\int_0^r s^2 g(s)\,ds \\ \int_0^r s^4 g(s)\,ds - \frac{1}{3}r^2\int_0^r s^2 g(s)\,ds \end{bmatrix},$$

$$\begin{bmatrix} e \\ g \end{bmatrix} = \begin{bmatrix} h_1(1) - h_1(r) - \frac{1}{2}r\int_0^1 sg(s)\,ds + \frac{1}{2}\int_0^r sg(s)\,ds \\ h_3(1) - h_3(r) - \frac{1}{3}r(1+r)\int_0^1 sg(s)\,ds + \frac{1}{3}(1+r)\int_0^r sg(s)\,ds \end{bmatrix},$$

and

$$\begin{bmatrix} f \\ h \end{bmatrix} = \begin{bmatrix} h_2(1) - h_2(r) - \frac{1}{2}r\int_0^1 s^2 g(s)\,ds + \frac{1}{2}\int_0^r s^2 g(s)\,ds \\ h_4(1) - h_4(r) - \frac{1}{3}r(1+r)\int_0^1 s^2 g(s)\,ds + \frac{1}{3}(1+r)\int_0^r s^2 g(s)\,ds \end{bmatrix}$$

Part (c) holds using the proof of Theorem 9 with $\begin{bmatrix} \frac{1}{2}\sigma_\varepsilon H_c(r) \\ \frac{1}{3}\sigma_\varepsilon J_c(r) \end{bmatrix}$ and

$$\begin{bmatrix} \frac{1}{2}\sigma_\varepsilon \left(H_c(1) - H_c(r) - r\int_0^1 \left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa) \right] ds + \int_0^r \left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa) \right] ds \right) \\ \frac{1}{3}\sigma_\varepsilon \left(J_c(1) - J_c(r) - r(1+r)\int_0^1 \left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa) \right] ds + (1+r)\int_0^r \left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa) \right] ds \right) \end{bmatrix}$$ replaced by

$$\begin{bmatrix} \frac{1}{2}\sigma_\varepsilon H_c(r) \\ \frac{1}{3}\sigma_\varepsilon J_c(r) \end{bmatrix} + \begin{bmatrix} h_1(r) \\ h_3(r) \end{bmatrix} + \begin{bmatrix} h_2(r) \\ h_4(r) \end{bmatrix}$$

and

$$\begin{bmatrix} \frac{1}{2}\sigma_\varepsilon \left(H_c(1) - H_c(r) - r\int_0^1 \left[W_c(s) + e^{sc}\widetilde{W_c}(\kappa) \right] ds + \int_0^r \left[W_c(s) + e^{sc}\widetilde{W_c}(\kappa) \right] ds \right) \\ \frac{1}{3}\sigma_\varepsilon \left(J_c(1) - J_c(r) - r\left(1+r\right)\int_0^1 \left[W_c(s) + e^{sc}\widetilde{W_c}(\kappa) \right] ds + \left(1+r\right)\int_0^r \left[W_c(s) + e^{sc}\widetilde{W_c}(\kappa) \right] ds \right) \end{bmatrix}$$

$$+ \begin{bmatrix} e \\ g \end{bmatrix} + \begin{bmatrix} f \\ h \end{bmatrix}$$

respectively. Then under the alternative

$$W_{cp}^{(T)}\left(k\right) \xrightarrow{d} \left[P_c\left(r\right) + \frac{4}{\sigma_\varepsilon^2} R_1\left(r\right) \right].$$

■

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