

## Problem Set 1: Solution

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I.

A. The economist behaves as if he solves

$$\max_{\{l_t\}} \sum_{t=0}^T \beta^t U(I(1-l_t, A), l_t), \quad (1)$$

where we have substituted  $I(e_t, A)$  for the reputation  $P_t$  and  $1-l_t$  for the level of effort  $e_t$ . The optimality condition for this optimization problem is

$$U_p(P_t) \cdot I_e(1-l_t, A) = U_l(l_t), \quad \text{for all } t \quad (2)$$

where  $U_l$  denotes the partial derivative of  $U$  with respect to the given argument and similarly for  $I_l$ .

B. Neither  $U(\cdot)$ , nor  $I(\cdot)$  depend directly on  $t$  hence the economist will exert the same level of effort throughout his career.

C. Let  $F(l, A) \equiv U_l - U_p I_e = 0$ . By the Envelope Theorem and the Implicit Function Theorem, we know that

$$\frac{dl}{dA} = -\frac{\partial F / \partial A}{\partial F / \partial l}. \quad (3)$$

Now,

$$\frac{\partial F}{\partial A} = -U_p I_{eA} \leq 0, \quad (4)$$

since  $I_{eA} \geq 0$ , while

$$\frac{\partial F}{\partial l} = U_{ll} + U_p I_{ee} - I_e U_{pl} \leq 0, \quad (5)$$

since  $U$  is concave in  $l$ ,  $I$  is concave in  $e$ , and  $U_{pl} = 0$ . Hence

$$\frac{dl^*}{dA} \leq 0$$

as long as  $I_{eA} \geq 0$ . High ability economists spend more time on research and less on leisure. Finally, since  $1 - l^*$  increases with  $A$  and reputation increases with both ability and effort, we have

$$\frac{dP^*}{dA} \geq 0,$$

so that high ability economists are more respected than their less able peers.

II.

D. The economist now behaves as if he solves

$$\max_{\{P_t, l_t\}} \sum_{t=0}^T \beta^t U(P_t, l_t), \quad (6)$$

$$\text{s.t. } P_t = (1 - \delta)P_{t-1} + I(1 - l_t, A) \quad \text{for all } t.$$

The Lagrange function for this problem is

$$\Lambda = \sum_{t=0}^T \beta^t U(P_t, l_t) + \lambda_t [(1 - \delta)P_{t-1} + I(1 - l_t, A) - P_t]. \quad (7)$$

To characterize the behavior of the economist, we want to analyze the first order condition of the Lagrange function with respect to reputation and leisure. The first order conditions are given by

$$[P_t]: \quad \beta^t U_p(t) - \lambda_t + \lambda_{t+1}(1 - \delta) = 0 \quad (8)$$

$$[l_t]: \quad \beta^t U_l(t) = \lambda_t I_e(t). \quad (9)$$

Notice that respect in time  $t$  influences utility both directly in time  $t$  and through its effect on reputation in the next period  $t + 1$ . Thus when the economist is deciding how much time to devote to research, the present as well as the future benefit of this research will be taken into account.

E. Use the first order conditions for leisure in periods  $t$  and  $t + 1$  to express  $\lambda_t$  and  $\lambda_{t+1}$  in terms of  $\beta, U_l$ , and  $I_e$ . Plugging these expressions into the first order condition for reputation and dividing both sides by  $\beta^t$ , we get

$$U_p(t) - \frac{U_l(t)}{I_e(t)} + \beta(1 - \delta) \frac{U_l(t+1)}{I_e(t+1)} = 0. \quad (10)$$

Now, in a steady state,  $P_{t+1} = P_t = P^*$  and  $l_{t+1} = l_t = l^*$ . Plugging these into the above equation and rearranging, we obtain

$$\frac{U_p(P^*)}{U_l(P^*)} = \frac{1 - \beta(1 - \delta)}{I_e(1 - l^*, A)}, \quad (11)$$

which says that the steady state values of reputation and leisure are such that the ratio of benefits (marginal benefits, the left-hand side) is equal to the ratio of shadow prices (marginal costs, the right-hand side).

F. Note that

$$P_t = (1 - \delta)P_{t-1} + I(1 - l_t, A). \quad (12)$$

We are looking for the locus of points where  $P_t = P_{t-1}$ . So  $\Delta P_t = 0$  implies that

$$P_t = I(1 - l_t, A)/\delta. \quad (13)$$

Similarly, using expression (10),  $\Delta l_{t+1} = 0$  implies that

$$U_p(P_t) = [1 - \beta(1 - \delta)] \frac{U_l(l_t)}{I_e(1 - l_t, A)}. \quad (14)$$

Clearly, as  $P_t$  increases,  $U_p$  declines. Hence for  $\Delta l_{t+1} = 0$  to be optimal,  $U_l/I_e$  needs to decline too, which requires an increase in  $l_t$ . Assuming that  $\lim_{l_t \rightarrow 0} U_l = +\infty$  and  $\lim_{l_t \rightarrow 1} I_e = +\infty$ , we obtain the following diagram (figure 1).

Why does the dynamical system behave in this fashion? Notice that at any point to the southeast of the  $\Delta P_t = 0$  curve, the economist exerts more effort than what is required to keep reputation constant. Since reputation is increasing in effort, for any point to the Southeast of  $\Delta P_t = 0$ , reputation will grow, while at any point to the Northwest of  $\Delta P_t = 0$ , reputation will decline. Similarly, note that at any point to the Southwest of the  $\Delta l_{t+1} = 0$  curve, the marginal utility of reputation is higher than what is required to keep effort constant. That is, the left-hand side of (14) is larger than the right-hand side. But then, the optimality condition (10) implies that the economist will exert more effort in the next period in order to equalize the marginal benefits and marginal costs. Hence at any point to the Southwest of the  $\Delta l_{t+1} = 0$  curve, effort is increasing, while at any point to the Northeast of  $\Delta l_{t+1} = 0$ , effort is declining.

Combining these, we see that the dynamical system is globally stable since for any initial condition, the transition path will eventually converge to the steady state.

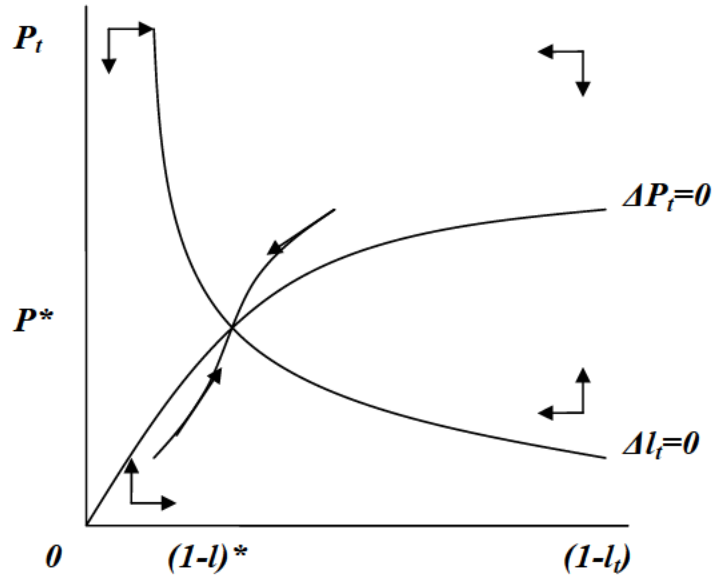


Figure 1: Phase diagram showing the transitional dynamics of the model. The steady state is globally stable. Any initial condition eventually leads to the steady state.

G. From (10) we know that the optimal behavior requires

$$U_p(P_0) = \frac{U_l(l_0)}{I_e(1-l_0, A)} - \beta(1-\delta) \frac{U_l(l_1)}{I_e(1-l_1, A)}. \quad (15)$$

So suppose  $P_0 \geq P^*$ . Then

$$P_0 = (1-\delta)P^* + I(1-l_0, A) - (1-\delta)(P_0 - P_{-1}), \quad (16)$$

which requires  $l_0 < l^*$  since  $(1-\delta)(P_0 - P_{-1}) \geq 0$ .

Now what happens if  $P_0 < P^*$ ? Then clearly

$$U_p(P_0) > U_p(P^*) = [1 - \beta(1 - \delta)] \frac{U_l(l^*)}{I_e(1 - l^*, A)}.$$

Suppose  $l_0 \geq l^*$ . Then  $l_1 \geq l^*$  and  $l_1 < l_0$ . Why? Look at the phase diagram. But then

$$\begin{aligned} & \frac{U_l(l_0)}{I_e(1 - l_0, A)} - \beta(1 - \delta) \frac{U_l(l_1)}{I_e(1 - l_1, A)} \\ & < [1 - \beta(1 - \delta)] \frac{U_l(l^*)}{I_e(1 - l^*, A)} < U_p(P_0). \end{aligned}$$

But then the particular sequence  $l_0, l_1$  cannot be an optimum. Hence we must have  $l_0 < l^*$ . We have thus shown that the economist will always work harder in the beginning of his career than in the steady state. Figure 2 shows a possible path for the case where the preferences and technology are such that  $P_0 < P^*$ .

H. Earnings start at a lower level than the steady state and are increasing during most of the economist's career, eventually reaching a plateau and even slightly decreasing at old age. Effort, on the other hand, starts at a level higher than the steady state, increases in the initial stages of the economist's career but then starts to decline earlier than reputation, even while reputation is still growing, and then keeps on declining with age.

I. Recall the optimality condition,

$$U_p(t) - \frac{U_l(t)}{I_e(t)} + \beta(1 - \delta) \frac{U_l(t+1)}{I_e(t+1)} = 0.$$

This condition will still hold for all  $t$  except  $T$ . For  $t = T$ , we now have

$$U_p(T) = \frac{U_l(T)}{I_e(T)}. \tag{17}$$

Aside from the last period, the dynamic properties of the model are thus not dramatically affected by the assumption that  $T$  is infinite.

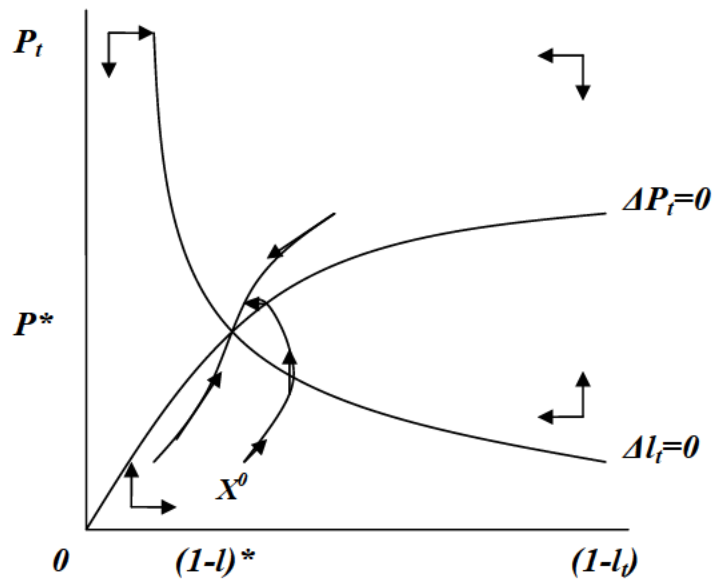


Figure 2: Transition path of reputation and effort given an initial condition of  $X^0$ .

ECN 601 Problem Set 2  
SOLUTION

In *A Treatise on the Family*, Gary Becker defined the following theorem, subsequently known as **The Rotten Kid Theorem**:

*Each beneficiary (in a household), no matter how selfish, maximizes the family income of his benefactor and thereby internalizes all effects of his actions on other beneficiaries.*

This problem set deals with the applicability (or lack thereof) of the Rotten Kid Theorem under various assumptions about the environment and production technology that households may encounter.

PART A. PUBLIC GOOD

Consider a household consisting of a patriarch  $P$ , endowed with some income  $Y$ , and two kids 1 and 2. There are two goods, one private  $x$  and one public  $Q$ , for which both prices are set to 1. Let  $U^i(x^i, Q)$  denote kid  $i$ 's utility, as a function of his private consumption  $x^i$  and the level  $Q$  of public goods. Also, let  $W(U^1, U^2)$  denote the patriarch's welfare index—for simplicity, we assume that the patriarch does not consume but only cares about the kids.

The kids play a two-stage game. At stage one, each kid decides on some amount  $q^i$  he will contribute towards the public good, with the total level being given by  $Q = q^1 + q^2$ . At stage two, taking  $Q$  as given, the patriarch uses the remaining household income  $Y - Q$  to buy the private good and redistribute it across the children, in order to maximize  $W$ . The question is whether the resulting allocation of the public good is efficient and optimal from the patriarch's point of view.

1. Write the program that characterizes efficient allocations of the public good. What are the corresponding first order conditions?

The Pareto efficient allocation solves

$$\begin{aligned} & \max_{\{x^1, x^2, Q\}} W(U^1(x^1, Q), U^2(x^2, Q)) \\ \text{s.t. } & x^1 + x^2 = Y - Q. \end{aligned}$$

The FOCs for  $x^1$  and  $x^2$  are

$$\frac{\partial W}{\partial U^1} \left[ \frac{\partial U^1}{\partial x^1} - \frac{\partial U^1}{\partial Q} \right] = \frac{\partial W}{\partial U^2} \frac{\partial U^2}{\partial Q}$$

and

$$\frac{\partial W}{\partial U^2} \left[ \frac{\partial U^2}{\partial x^2} - \frac{\partial U^2}{\partial Q} \right] = \frac{\partial W}{\partial U^1} \frac{\partial U^1}{\partial Q}.$$

Combining, we get

$$\frac{\partial U^1 / \partial Q}{\partial U^1 / \partial x^1} + \frac{\partial U^2 / \partial Q}{\partial U^2 / \partial x^2} = 1, \quad (1)$$

which is the well-known Lindahl condition.

2. Taking  $Q$  as given, write the program of the patriarch. Write its first order conditions.

At stage 2, taking  $Q$  as given, the patriarch redistributes the private commodity between the children. The resulting allocation solves

$$\begin{aligned} & \max_{\{x^1, x^2\}} W(U^1(x^1, Q), U^2(x^2, Q)) \\ \text{s.t. } & x^1 + x^2 = Y - Q. \end{aligned}$$

Eliminating  $x^2$ , the FOC is

$$\frac{\partial W}{\partial U^1} \frac{\partial U^1}{\partial x^1} = \frac{\partial W}{\partial U^2} \frac{\partial U^2}{\partial x^2}. \quad (2)$$

So the patriarch sets the ratio marginal utilities across children to the ratio of "prices" since the price of increasing child 1's welfare is  $1/(\partial U^1 / \partial x^1)$ .

3. Will the optimal distribution  $(x^1, x^2)$  chosen by the patriarch depend on the individual contributions  $(q^1, q^2)$  or only on the total quantity of the public good,  $Q = q^1 + q^2$ ?

Only on the total quantity. This is because (2) depends only on the total quantity of public good—kids' utilities depend on  $Q$ , not  $q^1$  and  $q^2$  separately.

4. At stage 1 the kids take the redistribution rule derived above as given. They understand that the patriarch will respond to any choice  $(q^1, q^2)$  by optimally redistributing the private commodity according to the rule derived in 3. Assume that the kids play simultaneously and non cooperatively. Write each kid's program.

At stage 1, each kid chooses his individual contribution to the public good,  $q^i$ , taking as given the redistribution scheme  $(x^1(Q), x^2(Q))$ . Each  $q^i$  must hence solve

$$\begin{aligned} & \max_{q^i} U^i(x^i(q^1 + q^2), q^1 + q^2). \\ \text{s.t. } & 0 \leq q^i \leq Y - q^j - x^1(Q) - x^2(Q). \end{aligned}$$

5. Assuming kid  $i$ 's program has an interior solution, write the first order condition.

If the solution is interior, the first order condition holds with equality, namely

$$\frac{\partial U^i(x^i(Q), Q)}{\partial x^i} \frac{\partial x^i(Q)}{\partial Q} + \frac{\partial U^i(x^i(Q), Q)}{\partial Q} = 0,$$

which gives

$$\frac{\partial x^i}{\partial Q} = - \frac{\partial U^i / \partial Q}{\partial U^i / \partial x^i}. \quad (3)$$

The chosen  $Q$  must solve this equation for both kids.

6. Assume both programs have an interior solution. Is the resulting allocation of private and public goods efficient and optimal from  $P$ 's viewpoint?

Differentiate the resource constraint with respect to  $Q$  to obtain

$$\frac{\partial x^1(Q)}{\partial Q} + \frac{\partial x^2(Q)}{\partial Q} = -1.$$

Given that both kids' problems have an interior solution, and substituting in the first order conditions that determine  $Q$ , we obtain

$$\frac{\partial U^1 / \partial Q}{\partial U^1 / \partial x^1} + \frac{\partial U^2 / \partial Q}{\partial U^2 / \partial x^2} = 1,$$

which is exactly the same condition as (1). Therefore the Rotten Kid Theorem applies. The outcome is optimal from the point of view of the patriarch because it is exactly the allocation he would choose if both  $x^i$  and  $Q$  were under his control.

7. Now assume that both kids are symmetric in the sense that: (i) utilities are identical,

$$U^1(x, Q) = U^2(x, Q) = U(x, Q) \quad \text{for any } (x, Q),$$

and (ii)  $W$  is symmetric,

$$W(U^1, U^2) = W(U^2, U^1) \quad \text{for any } (U^1, U^2).$$

Describe the solution to the game. Is it optimal?

Under symmetry there always exists an interior, egalitarian solution such that

$$x(Q) = x^1(Q) = x^2(Q) = \frac{Y - Q}{2}$$

and (3) boils down to an equation independent of  $i$ , namely

$$\frac{1}{2} \frac{\partial U}{\partial x} = \frac{\partial U}{\partial x}.$$

Then both kids contribute a positive amount and the outcome is efficient and optimal by the previous argument.

8. Now assume that preferences are given by

$$U^i(x^i, Q) = \log(x^i) + \alpha_i f(Q)$$

where  $f$  is a concave, strictly increasing function, and the patriarch's welfare function is

$$W(U^1, U^2) = \lambda^1 U^1 + \lambda^2 U^2.$$

Compute the outcome of the second-stage as a function of  $Q$ . Write the first order conditions for each kid's program. Are they compatible? What does that tell you about the scope of the result you obtained in part 6?

Now, the outcome of the second-stage game is given by

$$\frac{\lambda^1}{x^1} = \frac{\lambda^2}{x^2},$$

which implies that

$$x^1 = \frac{Y - Q}{1 + \lambda^2/\lambda^1}$$

and

$$x^2 = \frac{Y - Q}{1 + \lambda^1/\lambda^2}.$$

The first order conditions of the first-stage problem then are

$$\frac{1}{\alpha_1} = (Y - Q)f'(Q)$$

and

$$\frac{1}{\alpha_2} = (Y - Q)f'(Q).$$

So if  $\alpha_1 = \alpha_2$ , then both equations hold. Otherwise, they are not compatible. What is going on? In general, condition (3) is really two equations in one unknown,  $Q$ . Generically, a solution does not exist. The preceding example shows this explicitly. Therefore, the two kids' first order conditions can't be interior, so the efficiency result in 6 and hence the Rotten Kid Theorem does not apply generally.

## PART B. HOUSEHOLD PRODUCTION

Consider now a couple in which each member  $i$  allocates his (her) time across market work  $l^i$  (at some wage  $w_i$ ), domestic work  $t^i$ , and leisure  $1 - l^i - t^i$ . Labor income plus some fixed initial wealth  $y$  are used to buy consumption goods  $x_1, \dots, x_n$  at the given market prices  $p_1, \dots, p_n$ . These goods and domestic work are used as inputs for household production, which yields a private good  $z$  according to the technology

$$z = f(x_1, \dots, x_n, t^1, t^2).$$

After it is produced,  $z$  is allocated within the couple with member  $i$  receiving  $z^i$ , such that  $z^1 + z^2 = z$ . Member 1 (the rotten kid) is egoistic, with a utility function of the form,

$$U^1(z^1, 1 - l^1 - t^1) = z^1 + v^1(1 - l^1 - t^1).$$

Member 2 (the matriarch) is altruistic with her utility function given by,

$$W(U^1, U^2)$$

where

$$U^2(z^2, 1 - l^2 - t^2) = z^2 + v^2(1 - l^2 - t^2).$$

1. Show that all Pareto efficient allocations require the same level of labor supply, domestic work, and market purchases. Write the corresponding program.

This is the Pareto problem:

$$\begin{aligned} & \max_{\{l_i, t_i, z_i, x\}_{i=1,2}} \theta_1 u^1 + \theta_2 u^2 \\ \text{s.t. } & z^1 + z^2 = f(x, t_1, t_2), \\ & \sum p_k x_k = w_1 l_1 + w_2 l_2 + y. \end{aligned}$$

Substitute in the specific utilities to obtain

$$\begin{aligned} & \max_{\{l_i, t_i, z_i, x\}_{i=1,2}} \{\theta_1 [z^1 + v^1(1 - l^1 - t^1)] + \\ & \theta_2 W [z^1 + v^1(1 - l^1 - t^1), z^1 + v^1(1 - l^1 - t^1)]\} \\ \text{s.t. } & z^1 + z^2 = f(x, t_1, t_2) \quad [\lambda] \\ & \sum p_j x_j = w_1 l_1 + w_2 l_2 + y \quad [\mu]. \end{aligned}$$

The FOCs are:

$$\begin{aligned} -\theta_1 v_L^1 - \theta_2 W_1 v_L^1 + \mu w_1 &= 0 & [l^1] \\ -\theta_2 W_2 v_L^2 + \mu w_2 &= 0 & [l^2] \\ -\theta_1 v_L^1 - \theta_2 W_1 v_L^1 + \lambda \frac{\partial f}{\partial t^1} &= 0 & [t^1] \\ -\theta_2 W_2 v_L^2 + \lambda \frac{\partial f}{\partial t^2} &= 0 & [t^2] \\ \theta_1 + \theta_2 W_1 - \lambda &= 0 & [z_1] \\ \theta_2 W_2 - \lambda &= 0 & [z_2] \\ \lambda \frac{\partial f}{\partial x_j} - \mu p_j &= 0 & [x_j]. \end{aligned}$$

The goal is to show that labor and domestic supplies as well as market purchases  $\{x_j\}$  do not depend on the Pareto weights. We do this by eliminating  $\theta_1$  and  $\theta_2$  from the FOCs:

From FOCs 1 and 5, we get

$$\lambda v_L^1 = \mu w_1.$$

From FOCs 2 and 6, we get

$$\lambda v_L^2 = \mu w_2.$$

These conditions say that an optimal allocation equates the marginal rates of technical substitution between household and market work for each member of the household.

From FOCs 3 and 5, we get

$$v_L^1 = \frac{\partial f}{\partial t^1}$$

while from FOCs 4 and 6

$$v_L^2 = \frac{\partial f}{\partial t^2}.$$

The first two conditions in this new set in turn imply that

$$\frac{v_L^1}{v_L^2} = \frac{w_1}{w_2},$$

and similarly

$$\frac{\partial f / \partial x_j}{v_L^1} = \frac{p_j}{w_1} \quad \text{for all } j.$$

We thus have a system of  $4+n$  equations in the  $4+n$  unknowns  $l^1, l^2, t^1, t^2, x$ . Neither  $\theta_i$  appears in these equations so the solution cannot depend on  $\theta_i$ . What is the intuition? We have managed to separate productive efficiency (equating marginal products) and redistribution within the household (which will be influenced by  $\theta_i$ ). How is that possible? Note that nothing is said about  $z^1$  and  $z^2$  as those will depend on  $\theta_i$ .

2. For any given, possibly suboptimal,  $x_1, \dots, x_n, t^1, t^2, l^1, l^2$ , what is the form of the Pareto set in the  $(U^1, U^2)$  plane?

The Pareto frontier is made of

$$\begin{aligned} u^1 &= z^1 + v^1 \\ u^2 &= W[z^1 + v^1, z^2 + v^2]. \end{aligned}$$

Remember that utility is an ordinal concept so we can always apply the monotone transformation  $W^{-1}$  and rename utility

$$\bar{u}^2 \equiv W^{-1}(u^2, u^1) = z^2 + v^2.$$

Then, we get

$$\begin{aligned} u^1 &= z^1 + v^1 \\ \bar{u}^2 &= z^2 + v^2 = f - z^1 + v^2. \end{aligned}$$

Now sum the two to eliminate  $z^1$ . Given that all other variables are fixed on the Pareto frontier, we have

$$u^1 + \bar{u}^2 = f + v^1 + v^2.$$

The LHS is just a constant, so the Pareto frontier in  $(u^1, \bar{u}^2)$  plane is a line with slope -1. The initial Pareto frontier is not a line but it becomes linear under a suitable monotone transformation. Not all points on this frontier will be on the original frontier. For instance,  $u^1$  will always be important as it enters both agents' utilities, so outcomes with extremely small  $u^1$  will likely not be Pareto efficient.

3. Does the Rotten Kid Theorem apply? Why or why not?

Yes. The utilities of the two consumers are transferable through  $z$  so the Rotten Kid Theorem applies. The idea is that since the matriarch can control the allocation of  $z$ , it can implement any allocation on the Pareto frontier by manipulating how  $z^1$  depends on  $(l^1, t^1)$ . This can also be shown directly.

The matriarch's problem is

$$\begin{aligned} &\max_{\{l^2, t^2, x\}} W[z^1 + v^1(1 - l^1 - t^1), z^2 + v^2(1 - l^2 - t^2)] \\ \text{s.t. } & z^1 + z^2 = f(x, t^1, t^2) \\ & \sum p_j x_j = y + w_1 l^1 + w_2 l^2. \end{aligned}$$

Taking the FOCs and combining, we get

$$\begin{aligned} \frac{p_j}{w_2} &= \frac{\partial f / \partial x_j}{v_L^2}, \\ v_L^2 &= \frac{\partial f}{\partial t^2}, \end{aligned}$$

and

$$W_1 = W_2.$$

The last condition determines the function  $z^1 = g(l^1, t^1)$  that the matriarch offers the kid. The kid's problem is

$$\max_{l^1, t^1} g(l^1, t^1) + v^1(1 - l^1 - t^1).$$

Combining the FOCs, we obtain

$$v_L^1 = \frac{\partial g}{\partial l^1} = \frac{\partial g}{\partial t^1}.$$

If the matriarch chooses  $g$  so that

$$\frac{\partial g}{\partial t^1} = \frac{\partial f(x^*, t^1, t^{2*})}{\partial t^1}$$

and

$$\frac{\partial g}{\partial l^1} = \frac{\mu}{\lambda} w_1,$$

then the non-cooperative solution will be determined by the same equations as the efficient outcome. The variables  $l^1, l^2, t^1, t^2, x$  will be determined by the same  $4 + n$  equations as in the efficient case and  $z^1$  and  $z^2$  will be determined as necessary to make  $W_1 = W_2$  hold.