Urban comparative statics when commuting cost depends on income

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Received 22 February 2004

Abstract

This article investigates the comparative-static properties of an urban model where commuting cost is a function of income. In the standard urban model (Wheaton, W.C., 1974. A comparative static analysis of urban spatial structure. J. Econ. Theory 9, 223–237) commuting cost depends only on distance, but a more realistic formulation makes commuting cost per mile an increasing function of income. In contrast to the results based on the standard urban model, land rent at the central business district rises as income grows if the time cost of commuting is greater than the operating cost.

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Keywords: Commuting cost; Income; Rent gradient

1. Introduction

In the standard urban model, commuting cost depends only on distance. Wheaton (1974) assumed that commuting cost is a function only of distance, and that the elasticity of marginal commuting cost with respect to distance is less than 1. He found that

* I thank the anonymous referees for their helpful comments and suggestions.

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1051-1377/$ - see front matter © 2005 Elsevier Inc. All rights reserved.
doi:10.1016/j.jhe.2005.03.003
as income increases, land rent falls at the central business district (CBD) while it increases beyond a certain distance from the CBD. As a result, the city becomes larger.

While there is general agreement among urban economists that income is an important factor determining commuting cost, there has been no analytical study based on a general equilibrium model assuming commuting cost as a function of both distance and income. While some research assumes commuting cost as a function of distance and income, these analyses are either not based on a fully characterized urban model (partial equilibrium analysis) or focused on mode choice (De Salvo, 1977; De Salvo and Huq, 1996; Fujita, 1989; McDonald, 1975; Mills and Hamilton, 1984; Sasaki, 1990).\(^1\) De Salvo (1977) and De Salvo and Huq (1996) derived the impact of income on housing demand, but they used a partial equilibrium model. Sasaki (1990) made use of the standard urban model with commuting cost as a function of distance and income for the analysis of two households’ mode choice behavior. Sasaki investigates the effect of income on welfare and other urban characteristic variables. He demonstrates that an increase in income of the rich living in the suburban area can either increase or decrease the welfare of the poor living in the central area; it can also increase or decrease land rent at the center of the city. However, he does not explain under what conditions these effects occur, and the results of his analysis are not based on formal comparative static analysis.\(^2\) Arnott et al. (1978, hereafter AMW) analyzed the effect of changes in income on utility under a two-income group setting with a commuting cost function based on income and distance.\(^3\) However, AMW used a numerical simulation model and did not perform comparative statics with a general functional model.

This article investigates the urban comparative-static effects of a change in income under a new and plausible form of the commuting cost function. By making use of the standard urban model—often called an Alonso–Wheaton type model (Alonso, 1964; Wheaton, 1974)—with a commuting cost function that is a function of income and distance, this analysis yields somewhat different analytical results from those derived from the standard model. The most significant difference is that land rent at the CBD can increase as income rises. When a CES or a Cobb–Douglas utility function is used, the direction of changes in land rent at the CBD depends on the relative magnitudes of operating and time costs.

2. The model

The model used here is a variant of the standard urban model with a commuting cost function based on income and distance. The standard model can be distin-

\(^1\) Partial equilibrium analysis based on the bid-rent approach in urban economics can be viewed as a general equilibrium analysis under an open model assumption because utility level is fixed in both cases.

\(^2\) Sasaki (1990) focuses on the effect of income on the boundary (z) between two income classes. From the ambiguous effect of income on the boundary, he deduces changes in endogenous variables due to an increase in household income.

\(^3\) AMW uses a different commuting cost function, \(k = 0.01t(y – \text{constant})\). Their commuting cost function does not include operating costs or time costs.
guished from others in that: (i) utility is a function of a composite good and land consumption; (ii) no housing production is considered; and (iii) commuting costs are a function of distance. This paper changes only the third feature. I make use of a linear commuting cost function in distance and income as is done in Muth (1969, 1975), Yinger (1979), Mills and Hamilton (1984), Fujita (1989), LeRoy and Sonstelie (1983), and Sasaki (1990). All other features of the standard urban model are retained.

The following model is the simple general equilibrium model of urban economics. The utility function in (1) is assumed to be strictly quasi-concave and have two normal goods as arguments. Eq. (2) is the budget constraint; and Eq. (3) is the commuting cost, which is composed of operating cost and time cost. Eq. (4) is the land market clearing condition at distance \( x \) from the CBD:

\[
\begin{align*}
\text{(1)} \quad & u = U(z, l), \\
\text{(2)} \quad & y = z + r(x)l + kx, \\
\text{(3)} \quad & k = k_o + k_y y, \\
\text{(4)} \quad & N(x)l(x) = 2\pi x, 
\end{align*}
\]

In Eqs. (1)–(4), \( z \) is the composite good; \( l \) is the land consumption; \( y \) is the income; \( r(x) \) is the land rent; \( x \) is the commuting distance from the CBD; \( k \) is the marginal commuting cost, \( k_o \) is the operating cost, \( k_y \) is a parameter for time cost of commuting; \( N(x) \) population living at \( x \) distance from the CBD. Utility is parametric and gross income is fixed. The land rent function [whose derivatives are given in (5)] is derived by maximizing \( r \) subject to the constraints (1) and (2) (Wheaton, 1974). This maximization then yields \( r \) and \( l \) as functions of all of the parameters of the problem. Most partial derivatives are the same as those derived in Wheaton (1974) except for the first two:

\[
\begin{align*}
\frac{\partial r}{\partial y} &= \frac{1 - xk_y}{l} > 0, \quad \frac{\partial r}{\partial k_y} = \frac{-xy}{l} < 0, \quad \frac{\partial r}{\partial x} = \frac{-k}{l} < 0, \quad \frac{\partial r}{\partial k_o} = \frac{-x}{l} < 0, \\
\frac{\partial r}{\partial u} &= -\frac{1}{lu} < 0, \quad \frac{\partial l}{\partial x} > 0, \quad \frac{\partial l}{\partial y} < 0, \quad \frac{\partial l}{\partial k_o} > 0, \quad \frac{\partial l}{\partial k_y} > 0, \quad \frac{\partial l}{\partial x} > 0. 
\end{align*}
\]

Eqs. (6) and (7) characterize the equilibrium of a closed urban model. Eq. (6) is the condition that, at the border of the city \( b \), the land rent paid for residential use should be the same as the rent \( s \) for alternative uses. Eq. (7) presents the condition that \( N \) people should be housed in the city:

\[
\text{(5)}
\]

\[
\text{(6)}
\]

\[
\text{(7)}
\]

\[4\] To save space, only partial derivatives that are needed later in the analysis are presented.

\[5\] The model can be considered as a semi-closed model in the sense that the land rent goes to absentee landlords (see Pines and Sadka (1986) for further explanation).
3. Comparative static analyses

3.1. The effect of an increase in income on utility

As does Wheaton (1974), I first derive partial derivatives by fixing utility, and then solve for utility and city boundary using the boundary condition (6), and population condition (7).

By replacing \( x/l \) with a partial derivative and integrating (7) by parts (8) is derived

\[
bs - \int_0^b r(u, x, y, k_o, k_y) \, dx = -\frac{kN}{2\pi}.
\]

Differentiating (8) with respect to \( y \) yields (9)

\[
\int_0^b \left( \frac{\partial r}{\partial u} \frac{du}{dy} + \frac{\partial r}{\partial y} \right) \, dx = \frac{k_y N}{2\pi}.
\]

Rearranging (9) leads to (10)

\[
\frac{du}{dy} = \frac{\frac{k_y N}{2\pi} - \int_0^b \frac{\partial r}{\partial y} \, dx}{\int_0^b \frac{\partial r}{\partial u} \, dx} = \frac{\int_0^b \frac{1}{l} \, dx - \frac{k_y N}{\pi}}{\int_0^b \frac{1}{lu} \, dx}.
\]

**Proposition 1.** The effect of an increase in income on the utility level is positive, i.e. (10) is positive. Therefore, an increase in income unambiguously leads to a higher utility.

**Proof.** The necessary proof is to show that the numerator is positive. The basic procedure of proof is the same as shown in Sasaki (1987). Let \( \frac{N}{nb^2} = \frac{1}{7} \) or \( \frac{N}{\pi} = \frac{k^2}{7} \), where \( \frac{1}{7} \) is an area weighted average density. Define \( \frac{1}{b} \int_0^b \frac{1}{l} \, dx \) another measure of average density, i.e., unweighted average density. According to Sasaki (1987), \( \frac{1}{b} \int_0^b \frac{1}{l} \, dx > \frac{1}{7} = \frac{N}{nb^2} \). Therefore, \( bk_y \int_0^b \frac{1}{l} \, dx > \frac{kLN}{\pi} \). If \( bk_y \int_0^b \frac{1}{l} \, dx > \frac{kLNi}{\pi} \) is true, then \( \int_0^b \frac{1}{l} \, dx > \frac{kLN}{\pi} \) is true too because \( bk_y \) is always less than 1. (\( k_o + k_yy \) is less than \( y \)) since total commuting cost is less than gross income. Therefore, \( bk_y = \frac{y - bky}{y} < 1 \).

3.2. The effect of an increase in income on city boundary

The effect on the boundary of an increase in income can be obtained by differentiating (6) with respect to \( y \) and making use of (10). Simple manipulation yields (11).
\[
\frac{db}{dy} = \frac{(1 - bk_y) \left( \int_0^b \frac{1}{u_x} \, dx - \int_0^b \frac{1}{1 - bk_y} \, dx \right)}{ku_y \int_0^b \frac{1}{u_x} \, dx}
\]

\[
= \frac{(1 - bk_y) \int_0^b \frac{1}{u_x} \left( \frac{u_x}{u(x)} - \frac{1 - 2k_y}{1 - bk_y} \right) \, dx}{ku_y \int_0^b \frac{1}{u_x} \, dx}.
\] (11)

The sign of (11) depends on the terms of the integral in the numerator. It is unclear whether the sum of the two terms in the integral is positive, because \( \frac{u_x}{u(x)} \geq 1 \), while \( \frac{1 - 2k_y}{1 - bk_y} = \lambda \) can be either greater than or less than 1. When commuting cost is only a function of distance \( \lambda = 1 \), so in the traditional urban model \( \frac{db}{dy} \) is always positive and the size of the city increases as income rises. While the sign of (11) appears ambiguous, careful scrutiny leads to the conclusion that it is positive. This can be explained in three ways. First, a rise in income increases households’ disposable income, and households consume more of both goods. This leads to an increase in households’ welfare as derived above and an increase in aggregate demand for land in a city. As a result, the city size must grow. In the process of regaining equilibrium after the rise in income, commuting cost increases since city size grows and land rent may rise locally or globally. However, neither increased commuting costs nor increased land rent completely exhaust the increase in income.

Second, it can be formally shown that the size of the city grows as income grows. \( \frac{u_x}{u(x)} \) is equal to 1 when \( x = b \) and greater than 1 elsewhere. \( \lambda \) is greater than 1 where \( x \) ranges from 0 to \( b/2 \), while it is less than 1 where \( x \) lies between \( b/2 \) and \( b \). \( \frac{1}{l(x)} \) in (11) is the population density. Therefore, it is valid to consider (11) as the population density scaled by the relative marginal utility of the composite good less the population density scaled by \( \lambda \) over the range of the integral. Fig. 1 presents the sign of (11), and three population density functions can be drawn: \( D_0(x) \) is the

![Fig. 1. Population density functions: the effect on the size of the city.](image)
original population density function; $D_1(x)$ and $D_2(x)$ are scaled population density functions. In the traditional model where commuting cost is only a function of distance, the area under $D_0(x)$ is compared with the area under $D_1(x)$. As the latter is greater than the former, in the traditional model an increase in income causes the size of city to grow unambiguously.

When the commuting cost is a function of income and distance, the area under $D_2(x)$ is compared with that under $D_1(x)$. $D_2(x)$ must intersect $D_0(x)$ at the midpoint, $b/2$, since the scalar $\lambda$ becomes 1. Similarly, $D_1(x)$ meets $D_0(x)$ at $b$ where $u_z(b)/u_z(x)$ becomes 1. Furthermore at 0 distance, $d_1$ (the intercept of $D_1(x)$) is likely to be greater than $d_2$ (the intercept of $D_2(x)$) since $d_1$ is always greater than $d_2$ when the utility function is CES.\(^6\) It might be possible that $d_1$ is smaller than $d_2$ under other utility functions, but the area under $D_2(x)$ will still be smaller than the area under $D_1(x)$ since two curves cross each other somewhere near the CBD. Two scalars change monotonically as distance increases, and $\lambda$ is likely to be less than the relative marginal utility over the range of $x$. Thus, two scaled population functions never cross each other as shown in Fig. 1, and the area of $D_2(x)$ is always less than that of $D_1(x)$. Therefore, the size of the city grows as income increases even when commuting cost is a function of distance and income.

Third, when the utility function is specified, it can be formally proved that (11) is positive.

**Proposition 2.** When utility is either Cobb–Douglas or CES, as income grows the boundary of the city $b$ elongates. Put differently, the effect of an increase in income on the size of city is positive, i.e. (11) is positive.

**Proof.** This is proved by contradiction. In case of Cobb–Douglas and CES utility functions, $u_z(b) = \frac{u}{(y-b)}$ and $u_z(x) = \frac{u}{(y-kx)}$. Therefore

$$\frac{u_z(b)}{u_z(x)} = \frac{y - kx}{y - kb} = \frac{y - k_\beta y x - k_\alpha y}{y - k_\beta y b - k_\alpha b} = 1 - k_\gamma y - k_\beta y x/y \quad 1 - k_\beta y - k_\alpha b/b/y.$$

Let $1 - xk_\gamma y = k_\gamma$ and $1 - bk_\gamma y = k_\beta$. Assume $u_z(b)/u_z(x) < (1 - xk_\gamma y)/(1 - bk_\gamma y) = k_\gamma/k_\beta$. Then $\frac{u_z(b)}{u_z(x)} = \frac{k_\gamma - xk_\gamma y/k_\gamma}{k_\beta - bk_\gamma y/k_\beta} < 0 \Rightarrow \frac{k_\beta - bk_\gamma y/k_\beta}{k_\gamma - xk_\gamma y/k_\gamma} < 0 \Rightarrow k_\beta - bk_\gamma y/k_\beta < 0$. This becomes $\frac{k_\beta}{k_\gamma} < \frac{x}{y}$. However, it is known that $\frac{k_\beta}{k_\gamma} = 1 - xk_\gamma y/(1 - bk_\gamma y)$, and the former is greater than the latter.

**3.3. The effect of an increase in income on $r(0)$**

A striking analytical difference between the traditional model as presented in Wheaton (1974) and the model used here is the effect of changes in income on the rent–distance function. Land rent is a function of parameters: $u$, $y$, $x$, $k_\alpha$, $k_\beta$. Eq. (12) is derived by totally differentiating $r(x)$ with respect to $y$. Eq. (12) shows

\(^6\) When the utility function is a CES, at 0 distance $u_z(b)/u_z(x) = y/(y-kb)$ and $\lambda$ is $1/(1-bk_\gamma)$, and the former is greater than the latter.
the effect of an increase in income on the rent at the CBD. The result is the same as the one from the traditional model except there is a third positive term in the numerator, and hence the sign of (12) is ambiguous.

\[
\frac{dr(0)}{dy} = \frac{\partial r(0)}{\partial u} \frac{du}{dy} + \frac{\partial r(0)}{\partial y} = \frac{u_z(0) \int_0^b \frac{1}{lu(x)} \, dx - \int_0^b \frac{1}{u} \, dx + \frac{k_N}{\pi}}{l(0)u_z(0) \int_0^b \frac{1}{lu(x)} \, dx}.
\]  

(12)

If it were not there, the rent at the CBD would always fall because the second term in the numerator is greater than the first one in absolute value. Since the range of all nontrivial elements in the integral is the same, it is possible to transform (12) into (13)

\[
\frac{dr(0)}{dy} = \frac{\int_0^b \frac{1}{l(0)u_z(0)} \left( \frac{u_z(0)}{u_z(x)} - (1 - 2xk_y) \right) \, dx}{l(0)u_z(0) \int_0^b \frac{1}{lu(x)} \, dx}.
\]  

(13)

Fig. 2 is drawn based on (13). In a traditional model, \(1 - 2xk_y\) is 1 since \(k_y = 0\) and \(u_z(0)/u_z(x)\) is less than 1. Therefore \(D_1(x)\), the population density function scaled by \(u_z(0)/u_z(x)\), underlies \(D_0(x)\), the original population density function as shown in Fig. 2. Thus, the numerator of \(\frac{dr(0)}{dy}\) represented by the area under \(D_1(x)\) minus the area under \(D_0(x)\) is unambiguously negative, and in a traditional model at the CBD land rent must fall as income increases. However, under the above commuting cost function (3), \(1 - 2xk_y \leq 1\) and \(D_2(x)\), the population function scaled by \(1 - 2xk_y\), can lie either above or below \(D_1(x)\). If \(D_2(x)\) lies under \(D_1(x)\) (i.e., \(u_z(0)/u_z(x)\)) is greater than \(1 - 2xk_y\), then the difference in the area between \(D_1(x)\) and \(D_2(x)\) is positive, and land rent at the CBD increases as income increases. Fig. 2 represents this possibility.

**Proposition 3.** The effect of an increase in income on \(r(0)\) is ambiguous and the sign of (13) depends on the relative magnitudes of operating and time cost of commuting.

**Proof.** In the case of a CES or a Cobb–Douglas utility function, the numerator of (13) can be greatly simplified. The relative marginal utility of the composite good is \(\frac{u_z(0)}{u_z(x)} = \frac{y - (k_z - k_y)x}{y} = 1 - \frac{k_z}{y} - xk_y\). Substituting this into (13) leads to (14)

\[
\frac{dr(0)}{dy} = \frac{\int_0^b \frac{1}{l(0)u_z(0)} \left( k_y y - k_o \right) \, dx}{l(0)u_z(0) \int_0^b \frac{1}{lu(x)} \, dx}.
\]  

(14)

From (14), it can be noticed that the rent at the CBD depends on the relative magnitudes of operating cost and time cost. In a traditional model, time cost is zero, and hence the numerator of (14) becomes unambiguously negative. However, if time cost is greater than operating cost, the rent at the CBD must increase as income increases. This is because, when commuting cost is a function of income and distance, an increase in income directly increases commuting costs.

\(^7\) The new scalar, \(u_z(0)/u_z(x)\) is less than 1 over \(x\), since the marginal utility of the composite good increases with \(x\). This is because households consume land more and the composite good less as they move away from the CBD.
Commuting cost increases faster in the model of this paper than in the traditional model, and households’ willingness to bid more for land closer to the CBD will be intensified.

3.4. Summary of analyses

The results of comparative static analysis are summarized in Table 1. Table 1 shows that making use of a realistic commuting cost function does not change most of the qualitative results of the previous comparative static analysis.

Three of the four results have the same signs. One significant difference is the effect of income on the rent at the CBD. When a commuting cost function is a function of income and distance, land rent at the CBD can either increase or decrease as income increases. A rapid increase in time cost due to an increase in income makes the rent–distance function shift up rather than rotate. Land rent at the CBD can either rise or

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fall depending on the response in the utility level as income rises. As income increases, utility level increases and the city expands. Also, land demand in the central area is likely to increase.8

4. Conclusion

This article investigates the comparative-static properties of an urban model where commuting cost is a function of income. In the standard urban model (Wheaton, 1974) commuting cost depends only on distance, but a more realistic formulation makes commuting cost per mile an increasing function of income. In contrast to the results based on the standard urban model, land rent at the central business district (CBD) rises as income grows if the time cost of commuting is greater than the operating cost. The effects of income on the boundary of the city and on utility are similar to those derived from the traditional urban model, but their magnitudes are reduced.

References


8 The comparative static analysis to find the effect of a change in income on land demand at the CBD is suppressed because, given that the impact of \( y \) on \( r(0) \) is ambiguous, the impact on \( l(0) \) is also ambiguous. The comparative static analysis is available from the author upon request.