Housing and Commuting:
The Theory of Urban Residential Structure

an e-book
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Version 1.1: September 2008

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Chapter 1.3: The Basic Urban Model
April 2005

1. Introduction

This chapter presents the basic model that forms the foundation of the field of urban economics. After discussing the model's assumptions, the chapter solves the model step by step, derives comparative static results, and discusses evidence on the model's key predictions. A series of exercises suitable for Ph.D. students in urban economics is also included.

This basic urban model contains the central features of the models developed by Alonso (1964), Muth (1969), and Mills (1967, 1972). It is based on a series of strong assumptions about the various markets that influence urban structure. These assumptions, which are presented in the next section, have served as an agenda for the field of urban economics in the sense that scholars have searched for models with less restrictive versions of each assumption. A list of selected studies that weaken these assumptions is provided at the end of the chapter.

Urban models have two key features. The first is that they are built around a household maximization problem in which a household decides how much housing to consume and where to live. This problem is presented in Section 3. The key new analytical tool that emerges from this problem is the

bid function, which indicates the amount a household is willing to pay for housing at every location. Bid-functions are derived in

Section 4. Another important element of an urban model, housing production, is introduced in Section 5, and competition between housing and other uses of land is considered in Section 6.

The second key feature is that urban models inevitable are general equilibrium models in the sense that they involve more than one market. The basic source of an urban model can be described in a simple sentence: Households care about where they live because they must commute to work. This simple statement contains elements of six markets: housing, because people live in housing; land and capital, because housing is built on land using capital; transportation, because people must use some form of transportation to get to work; labor, because people are commuting to a job; and the market for some unstated product that is exported from the urban area, because a firm will not hire people unless it has a product to sell. Thus an urban model is not complete unless it includes some analysis, however rudimentary, of each of these six markets, that is, unless it is a general equilibrium model.

This chapter begins by focusing on the basic household maximization problem. The models in the first six sections are all incomplete in the sense that they do not ensure either that all available land is used or that there is enough housing for all urban workers. To put it another way, they are incomplete because they do not ensure equilibrium in the land and labor markets. Equilibrium conditions for these markets are introduced in Section 7, resulting in a **complete** urban model. Two types of complete urban model have appeared in the literature: **closed** models rule out migration between urban areas, whereas **open** models assume that migration is costless. The distinction between open and closed models is explored in Section 7, a closed model is solved in Section 8, and an open model is solved in Section 9.

The last three substantive sections in this chapter examine the conclusions that can be drawn from urban models. Section 10 shows how population density and building height vary with location; Section 11 shows how the results of the model change as key parameters change, that is, it presents some comparative static analysis; and Section 12 compares the predictions of the model with some evidence about actual urban residential structure. A series of exercises is provided at the end of this chapter.

2. Key Assumptions

This section presents the multi-part assumptions that form the foundation of the standard urban model. As noted earlier, each of the assumptions is quite restrictive, and each one has been relaxed in the literature. See the references at the end of this chapter.

Assumption 1: Housing Demand. Household utility functions depend on a composite consumption good and housing and take the Cobb-Douglas form.

Let U stand for utility, Z measure the consumption of the composite good, and H measure the consumption of housing. As explained more fully below, H is expressed in units of **housing services.** This assumption indicates that the utility function can be written $\hat{U} = \alpha_1 \ln\{Z\} + \alpha_2 \ln\{H\}$, where α_1 and α_2 are constants and "ln" indicates a natural logarithm. Remember that a household's decisions are invariant to any monotonic transformation of its utility function.[1] Equivalent expressions for this utility function, therefore, are $\tilde{U} = (1-\alpha)\ln\{Z\} + \alpha \ln\{H\}$, where $\alpha = \alpha_2 /(\alpha_1 + \alpha_2)$, and $U = Z^{(1-\alpha)} H^{\alpha}$. [2]

A note on notation: Throughout this chapter, curly brackets, {and}, are used to enclose the argument of a function, whereas parentheses and square brackets are used to clarify algebraic expressions. The expression $\ln\{Z\}$, for example, indicates that Z is the argument of the ln function.

Assumption 2: Housing supply. Housing services are produced with land and capital according to a Cobb-Douglas production function with constant returns to scale, and housing is owned by absentee landlords.

The first part of this assumption can be expressed by the equation $H = K^{1-a} L^{a}$ where K stands for capital and L stands for land. This production function focuses on the production of housing services from existing structures, which require little labor input. Note that this approach ignores the role of maintenance, rehabilitation, and conversion activities in the provision of housing.

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[1] This result is proved in most microeconomic texts

[2] Note that the symbols "^" and "~" are used to distinguish among various forms of the utility function. For convenience, the last form has no distinguishing symbol.

Assumption 3: The Transportation Network. Households commute between place of residence and place of work in a straight line at a constant transportation cost per mile using a single transportation mode.

The traditional phasing of this assumption has been that the urban area is located on a featureless plain. In addition, however, most urban models ignore any complexities in commuting patterns that arise because of the character of the transportation network. There is no distinction between commuting arteries and side streets, between radial streets and a street grid, or between various transportation modes (such as cars, buses and subways), and traffic congestion is not considered. Instead, the implicit transportation network is an abstract one in which everyone travels along the shortest distance between home and work at a constant cost per mile.

Assumption 4: Why Location Matters. In selecting a place to live, distance to work is the only locational characteristics households care about.

This assumption rules out neighborhood amenities. Households are assumed not to care about access to shopping or recreation, or about air quality, about the characteristics of their neighbors, or about any other feature of the location where they live.

Assumption 5: Types of Households. All households are alike.

According to this assumption, all households have the same income, family composition, and utility function.

Assumption 6: Labor Market. Income is fixed and all households have a single worker with a job in the central business district (CBD).

These assumptions deal with the labor market. The first part of this assumption, which is built on an implicit view of the labor market that is discussed in Section 7, greatly simplifies the analysis of an urban model. Combined with Assumption 2, the second part of this assumption implies that a household's commuting cost depends only on the distance between its residence and the CBD.

Assumption 7: Household Mobility. All households are perfectly mobile within an urban area; that is, if they have an opportunity to improve their utility, they will take it.

This assumption is fundamental to the logic of urban models. As we will see, it implies that all households achieve the same utility level.

Assumption 8: Local Governments. There are no local governments.

The standard urban model leaves out the local government sector.

3. The Basic Household Maximization Problem

An urban model is built around a household maximization problem in which the household decides how much housing and how much of the composite good to consume, as well as where to live. This section presents this problem with general utility and transportation-cost functions; Section 4 derives specific solutions with the functions in Assumptions 1 and 2.

The two goods in a household's utility function are a composite good and housing. The composite good, labeled Z, is measured in units that have a price of unity. It consists of all consumption goods except housing.

Housing is measured in units of housing services, H, which are the annual services provided by a house or apartment. The quantity of housing services in a given housing unit is a function of the characteristics of that unit, including its size, number of rooms, quality of construction, and so on. The basic urban model presented here, like most urban models, focuses on housing services and does not explicitly consider housing characteristics.

The annual price per unit of housing services is P. As we will see, this price depends on household location, measured in miles from the CBD, u; that is, $P = P\{u\}$. We will also see that $P'\{u\}$, the derivative of $P\{u\}$ with respect to u, is negative. In other words, the price of housing services is lower at greater distances from the CBD.

Note that throughout this chapter, a prime is used to indicate the derivative of a single-valued functions; for example $P'\{u\} = dP\{u\} / du$

Although, according to Assumption 2, the models in this chapter focus on the case of renters, this approach to housing can be applied to owners or renters. In the rental market, an observed apartment rent equals the housing services in that apartment multiplied by the price per unit of housing services at that location. In the sales market, we observe the price of a house, which is an asset, not the annual value of the services from that asset, which is the implicit rent. However, the amount someone would pay for a house that is expected to yield H units of housing services per year at price $P\{u\}$ per unit is the present value of the stream $P\{u\}H$ over the lifetime of the house. Let V stand for the market value, let i be the household's real discount rate, and let M be the house's expected lifetime. Then the appropriate present value expression is: [3]

$$V = \sum_{y=1}^{M} \frac{P\{u\}H}{(1+i)^{y}}. \tag{1}$$

Because summations are difficult to work with, it will prove convenient to use an alternative formulation of this equation, namely: [4]

$$V = \frac{P\{u\}H}{i^{*}}, \tag{2}$$

where

$$i^{*} = \frac{i}{1-(1+i)^{-M}}. \tag{3}$$

[3] This expression leaves out several complications, including the tax treatment of owner-occupied housing, mortgage terms and conditions, risk and uncertainty. It implicitly assumes static expectations and requires a real interest rate. For more on these issues, see Yinger et al. (1988).

[4] To derive equation (2), simply divide equation (1) by (1+ i) and subtract the result from equation (1).

Note that i^* can be interpreted as the **infinite-horizon discount rate,** that is, as the discount rate assuming an infinite horizon that is equivalent to a discount rate i with a horizon M. If M is large, above about 40 years, then i^* is a close approximation for i, and (2) simplifies to:

$$V = \frac{P\{u\}H}{i}. \tag{4}$$

Moreover, the annual flow associated with this asset value is simply $iV = P\{u\}H$, which is the same as apartment rent. Under some conditions, namely no down payment, no taxes, and a mortgage rate equal to the opportunity cost of investing in housing, this annual flow is observed in the form of the household's annual mortgage payment.

The worker in a household also must commute to work. Let $T\{u\}$ be the annual round-trip cost of commuting from residential location u to the CBD. A more distant location always implies greater commuting costs, so that $T'\{u\} > 0$.

On the basis of this discussion, the problem facing a household with daily income Y is to select Z, H, and u so as to:

Maximize $\quad U\{Z,H\}$

Subject to: $\quad Y = Z + P\{u\}H + T\{u\}. \quad (5)$

The Langrangian expression for this problem is:

$$\text{L} = U\{Z,H\} + \lambda\left(Y - Z - P\{u\}H - T\{u\}\right), \tag{6}$$

where λ is the Langrangian multiplier.

The first-order conditions are:

$$\frac{\partial U}{\partial Z} - \lambda = 0, \tag{7}$$

$$\frac{\partial U}{\partial H} - \lambda P\{u\} = 0, \tag{8}$$

and

$$\lambda\left(P'\{u\} H + T'\{u\}\right) = 0. \tag{9}$$

Combining (7) and (8) leads to a standard result in microeconomics:

$$\frac{\partial U / \partial H}{\partial U / \partial Z} = P\{u\}. \tag{10}$$

In words, the marginal rate of substitution between H and Z must be set equal to the ratio of the price of H to the price of Z.

It will prove convenient to develop an alternative interpretation of this condition. In a household maximization problem, the Lagrangian multiplier associated with the budget constraint, λ, can be interpreted as the marginal utility of income.[5] According to Equation (7), therefore, the household sets $\partial U / \partial Z$ equal to the marginal utility of income. Thus, the left side of (10) is the marginal utility of H divided by the marginal utility of income, which is the same as the marginal value of H in dollar terms. Another interpretation of (10), therefore, is that the household sets the marginal dollar benefit from H equal to its marginal dollar cost.

The third condition is not found in standard microeconomics texts, however. It is the condition that introduces us to the logic of urban models. The Lagrangian multiplier obviously can be canceled leaving:

$$P'\{u\} H + T'\{u\} = 0. \tag{11}$$

The first term is the marginal change in the cost of housing, at the value of H selected by the household, from moving one mile farther from the CBD. The second term is the increase in transportation cost from moving one mile farther from the CBD. Since $T'\{u\}$ and H are positive, this condition can only be satisfied at a point where $P'\{u\}$ is negative, that is, where $P\{u\}$ declines as one moves away from the

CBD. More specifically, it is satisfied at the location where the housing cost savings from moving a little farther out is just offset by the transportation cost increase.

Now comes the twist that drives urban models, which is the fundamental insight provided by Alonso (1964). According to Assumptions 5 and 7, all households are alike and perfectly mobile. If one location is best for one household, it also is best for every other household. However, everyone obviously cannot live in the same spot. Households compete for housing in a desirable location, driving up the price there. This competition ends only when the price has been driven up enough so that no household prefers that location to any other.

The same competition occurs at every location in the metropolitan area. **Locational equlibrium** exists only when no household has an incentive to move, that is, only when households are indifferent across locations. Because Equation (11) is a condition for household equilibrium, this condition must hold at every location. Thus, another way to state the locational equilibrium condition is that $P\{u\}$ must satisfy the equation:

$$P'\{u\} = \frac{-T'\{u\}}{H}. \qquad (12)$$

A $P\{u\}$ function that satisfies this equation indicates the maximum amount a household is willing to pay at one location relative to another and is therefore known as the equilibrium **bid function.**

As written, Equation (12) is a differential equation, the solution to which, namely $P\{u\}$, depends on specific forms for $T\{u\}$ and the utility function. If differential equations are not your cup of tea, don't worry. As shown in Section 4, you do not have to solve a differential equation to find $P\{u\}$. Section 4 also shows, however, that given Assumptions 1 and 3, this differential equation can easily be solved directly.

[5] Again, see a microeconomics text.

Before we find a specific solution for $P\{u\}$, it is worth noting that two of the properties of this bid function follow directly from Equation (12). First, because $T'\{u\}$ and H are positive, $P'\{u\}$ must be negative, that is, housing bids must be a downward-sloping function of distance. Consumers will substitute toward H as its price drops, so that housing consumption will increase with the distance from the CBD. Because $T'\{u\}$ is constant by Assumption 3, it follows that the second property of $P\{u\}$ is that its slope declines in absolute value as one moves away from the CBD. In other words, substitution between housing and the composite good insures that the second derivative of $P\{u\}$ is positive. Figure 1 presents an example of a bid function with these two properties.

Finally, note that the equilibrium defined by Equation (12) does not require any particular household to be in any particular location. In fact, an urban model does not determine where a household lives; instead, it determines what the price of housing would have to be for the people who live at each location to be content to stay there. Something outside the model, such as idiosyncratic preferences for housing or location, must control which households end up at each location.[6]

4. Locational Equilibrium or Deriving Bid Functions

Now let us derive a specific bid function using the assumptions in Section 2. The Cobb-Douglas utility function in Assumption 1 takes the form[7]:

$$\tilde{U} = (1-\alpha)\ln\{Z\} + \alpha\ln\{H\}. \tag{13}$$

It follows that

$$\frac{\partial \tilde{U}}{\partial Z} = \frac{1-\alpha}{Z} \tag{14}$$

[6] In a multi-class urban model, the model can determine the set of locations in which a class locates, but it cannot determine which household within a class lives in which of the locations allocated to that class.

and

$$\frac{\partial \tilde{U}}{\partial H} = \frac{\alpha}{H}. \tag{15}$$

Moreover, according to Assumption 3, a household must pay a constant cost per mile, t, to commute to work. If a household lives u miles form the CBD, therefore, its $T\{u\}$ equals tu. This t should be interpreted as the daily round-trip cost, which includes both the operating costs (for running a car or paying bus fare) and the opportunity cost of time spent commuting, which depends on travel speed, MPH, and the valuation of travel time as a fraction of the wage, w. In symbols,

$$t = t_0 + t_y Y, \tag{16}$$

where t_0 stands for round-trip operating costs per mile and t_y stands for round-trip time costs as a fraction of income. Moreover, the hourly wage rate is *Y/8* and the time it takes to travel one mile is 1/MPH, so round-trip time costs can be written as $2w(Y/8)(1/MPH) = wY/(4MPH)$.

We can now derive the demand functions for Z and H. First, substitute the two marginal utility expressions, (14) and (15), into the first-order conditions (7) and (8). Then rearrange the two conditions to isolate λ on one side, equate the two expressions for λ, and solve the resulting equation for Z. These steps yield $Z = \left[(1-\alpha)/\alpha\right]P\{u\}H$. Substituting this result into the budget constraint makes it possible to solve for the demand function for H, which is:

$$H = \frac{\alpha\left(Y - tu\right)}{P\left\{u\right\}}. \tag{17}$$

The demand function for Z follows directly:

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[7] A note on notation. In equations (5) through (10) and the accompanying text, "*U*" stands for a general utility function. Elsewhere, "*U*" or "$\tilde{U}$" stand for specific utility functions, such as equation (13).

$$Z = (1-\alpha)(Y - tu). \tag{18}$$

The bid function, $P\{u\}$, can now be found in two different ways, both of which will be presented. The two approaches are both worth examining because each of them is convenient for solving some types of more-complicated models.

4.1. The Indirect Utility Function Approach

The first method, which was popularized by Solow (1972), makes use of a concept known as an **indirect utility function.** The basic idea of an indirect utility function is straightforward. Direct utility functions depend on quantities of commodities. By substituting the demand functions for these commodities into the direct utility function, one can obtain utility as a function of income and prices, which is an indirect utility function. For a general discussion of this concept, see a microeconomics text.

Substituting the demand functions, (17) and (18), into the direct utility function yields the indirect utility function for our case. Remember that household choices are invariant with respect to monotonic transformations of the utility function. To simplify the algebra, therefore, we apply the exponential transformation to the direct utility function (13). Thus, the indirect utility function is:

$$U = \frac{k(Y - tu)}{P\{u\}^{\alpha}}, \tag{19}$$

where

$$k = (1-\alpha)^{1-\alpha}\,\alpha^{\alpha} \tag{20}$$

and U is the level of utility.

Because households are alike and mobile (by Assumptions 5 and 7), U must take the same value, say U^*, for every household. We cannot determine what this level is without further assumptions, but we know that it is the same for all households, that is, it is a constant. Thus, we can solve (19) for the bid function, $P\{u\}$. The result:

$$P\{u\} = \left(\frac{k}{U^*}\right)^{1/\alpha} \left(Y - tu\right)^{1/\alpha} \tag{21}$$

or

$$P\{u\} = \gamma\left(Y - tu\right)^{1/\alpha}, \tag{22}$$

where γ is an unknown constant. In a later section we will solve for this constant, which is equivalent to solving for the utility level, U^*. One might say that (21) gives the general shape of the bid function, but does not reveal its height. In fact, there exists a family of bid functions, each member corresponding to a different utility level. As illustrated in Figure 2, higher utility levels correspond to lower bid functions. This link has strong intuitive appeal; higher housing prices use up more of a household's resources and, with a constant income, lead to lower utility.

4.2. The Differential Equation Approach

The second approach, which was first used by Alonso (1964), Mills (1967, 1972), and Muth (1969), is to interpret Equation (12) as a differential equation and solve it for $P\{u\}$. In the case of Cobb-Douglas utility functions, this derivation is straightforward.

The derivation begins by substituting the transportation cost function, $T\{u\} = tu$, and the demand for housing, Equation (17), into the locational equilibrium condition, (12). These steps yield:

$$P'\{u\} = \frac{-tP\{u\}}{\alpha\left(Y - tu\right)} \tag{23}$$

Or

$$\frac{P'\{u\}}{P\{u\}} = \frac{-t}{\alpha\left(Y - tu\right)}. \tag{24}$$ Equation

(24) is known as an exact differential equation because all the expressions involving $P\{u\}$ are on the left side and all the expressions involving u are on the right side. Exact differential equations can be solved by integrating each side separately. In this case, both sides take the same general form, namely $g'\{u\}/g\{u\}$. On the left side, the g function is simply $P\{u\}$ and on the right side it is $(Y - tu)$. Thus, both sides of the equation can be solved with the following basic integral:

$$\int \frac{g'\{u\}}{g\{u\}} du = \ln\left\{g\{u\}\right\} + \kappa, \tag{25}$$

where κ is a constant of integration.

Applying (25) to both sides of (24), we obtain:

$$\ln\left\{P\{u\}\right\} = \frac{1}{\alpha}\ln\left\{Y - tu\right\} + \mathrm{K}. \tag{26}$$

Applying the exponential function to both sides of this equation yields the bid function:

$$P\{u\} = e^{\kappa}\left(Y - tu\right)^{1/\alpha} \tag{27}$$

or

$$P\{u\} = \gamma\left(Y - tu\right)^{1/\alpha}, \tag{28}$$

where $\gamma = e^{\kappa}$ is an unknown constant. This is, of course, exactly the same result we obtained using an indirect utility function.

5. Housing Production or Bringing in Land

The next step in developing an urban model is to add housing production. With this addition, we can determine the pattern of land prices in an urban area.

According to Assumption 2, housing services are produced with land and capital. Although labor obviously is an important input in the building of houses and apartments, it is not a major input in the production of housing services from existing dwellings and is not considered a housing input in most urban models. With the Cobb-Douglas form of Assumption 2, the quantity of housing services produced at location $u, H_s\{u\}$, is given by:

$$H_s\{u\} = AK\{u\}^{1-a} L\{u\}^a, \tag{29}$$

where $K\{u\}$ is the amount of capital used for housing at location u, $L\{u\}$ is the square miles of land available for housing at location u, and a and A are constants.

Profit-maximizing housing firms set the value of the marginal product of each input equal to its price. Because the production function has constant returns to scale, payments to factors by profit-maximizing firms exactly exhaust total revenue and the firm earns zero economic profits. The marginal products are found by differentiating Equation (29), so the conditions for profit maximization can be written:

$$\frac{(1-a)P\{u\}H_s\{u\}}{K\{u\}} = r \tag{30}$$

and

$$\frac{aP\{u\}H_s\{u\}}{L\{u\}} = R\{u\}, \tag{31}$$

where r is the annual rental rate for a unit of capital and $R\{u\}$ is the rent per acre of land at location u.

With a national capital market, the rental rate for capital does not depend on location and is not influenced by events in one metropolitan area. We assume, therefore, that r is constant.

In contrast, $R\{u\}$ is endogenous to the model. This is a key point. Remember that factors receive the value of their marginal product, total revenue is just exhausted, and the firm earns zero profits. Because the price-distance function is exogenous to firms, Equation (31) indicates that land rent adjusts so that land is paid its marginal product—and firms earn zero profits—at every location. Thus, the form of the rent-distance function is determined by the form of the price-distance function—not vice versa. It is not correct to say that housing prices are high in a given location because land is expensive there. Exactly the opposite is true: land rents are high in a given location because housing is expensive there. To put it in more general terms, land rents are high at a particular location because people are willing to pay a great deal there for housing or for some other economic activity that requires land.

Now we can solve Equation (30) for $K\{u\}$ and (31) for $L\{u\}$ and substitute the results into the housing production function, (29). These steps yield:

$$\begin{aligned} H_s\{u\} &= A\left(\frac{(1-a)P\{u\}H_s\{u\}}{r}\right)^{1-a}\left(\frac{aP\{u\}H_s\{u\}}{R\{u\}}\right)^{a} \\ &= Aa^a\left(\frac{1-a}{r}\right)^{1-a}\left(\frac{P\{u\}H_s\{u\}}{R\{u\}^a}\right). \end{aligned} \tag{32}$$

The supply of housing services, $H_s\{u\}$, can be canceled from both sides of this equation. After this cancellation and multiplying both sides by $R\{u\}^a$, this equation becomes:

$$R\{u\}^a = CP\{u\}, \tag{33}$$

where

$$C = Aa^a\left(\frac{1-a}{r}\right)^{1-a}. \tag{34}$$

With a little algebra, Equation (33) can be rewritten to express $R\{u\}$ as a function of $P\{u\}$, or vice versa:

$$R\{u\} = \left(CP\{u\}\right)^{1/a} \tag{35}$$

or

$$P\{u\} = \frac{R\{u\}^{a}}{C}. \tag{36}$$

This relationship between rents and housing prices is at the heart of an urban model.

By combining (35) with (22), we can now determine how land rents vary with location. In particular,

$$R\{u\} = C^{1/a}\left(\gamma\left(Y - tu\right)^{1/\alpha}\right)^{1/a} = \gamma^{*}\left(Y - tu\right)^{1/a\alpha}, \tag{37}$$

where γ^{*} is an unknown constant.

This equation, which describes the amount housing producers bid for land, is also known as a bid function. To distinguish between bid functions for housing and land, it is common practice to refer to Equation (22) as the **bid-price function** and Equation (37) as the **bid-rent function.**

6. Anchoring Bid Functions

Equations (22) and (37) contain unknown constants, γ and γ^{*}. In this section we take the first step toward "anchoring" the bid functions, that is, solving for these constants. This step is equivalent to selecting one bid function out of the family of bid functions described by each of these equations.

This step is based on the observation that housing must compete with other activities for the use of land. Although many non-housing activities exist in an actual urban area, a standard urban model assumes that there is one urban activity, namely housing, and one non-urban activity, namely agriculture. Now suppose that the value of land in agriculture, $\overline{R}$ per square mile, is constant around the urban area. Then housing will only exist where $R\{u\}$ is greater than $\overline{R}$. In other words, the urban area extends out to the location $\overline{u}$ at which

$$R\{\overline{u}\} = \overline{R}. \tag{38}$$

This condition can be used to anchor Equation (37), that is, to solve for γ^*. Evaluating (37) at $\overline{u}$ and setting the result equal to $\overline{R}$ yields:

$$\gamma^* = \frac{\overline{R}}{C^{1/a}\left(Y - tu\right)^{1/a\alpha}} \quad , \tag{39}$$

Combining this result with (37) leads us to the "anchored" form of the bid-rent function:[8]

$$R\{u\} = \overline{R}\left(\frac{Y - tu}{Y - t\overline{u}}\right)^{1/a\alpha} \quad . \tag{40}$$

Now, using Equation (40), we can also write down an "anchored" version of the bid price function. To be specific, combining (36) and (40) leads to:

$$P\{u\} = \overline{P}\left(\frac{Y - tu}{Y - t\overline{u}}\right)^{1/\alpha} \quad , \tag{41}$$

where

$$\overline{P} = \frac{\overline{R}^a}{C}. \tag{42}$$

In this equation, $\overline{P}$ can be interpreted as the opportunity cost of investing in housing.

The observant reader will note that the analysis in this section literally anchors the bid functions only if $\overline{u}$ is known. All bid functions describe locational equilibrium within an urban area, but as we will discover in the following sections, only an anchored bid function describes an equilibrium

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[8] Note that one can obtain exactly the same result by substituting Equation (21) instead of (22) into (33). In this case, one must solve for U^* instead of for γ^*.

across urban areas. To emphasize this second equilibrium condition, we will say that the anchored bid-price function is the **price-distance function** and the anchored bid-rent function is the **rent-distance function.**

If $\bar{u}$ is unknown, this analysis simply provides an alternative formulation of the constant term in Equation (22). A higher value for $\bar{u}$ corresponds to a higher intercept for the bid function and hence, according to Equation (21), to a lower level of household utility. Intuitively, the longer the longest commute, the lower the net income of a household at the outer edge of the urban area—and hence, the lower the utility of all the other (identical!) urban residents. To derive $\bar{u}$, we must specify a "complete" urban model. We now turn to this task.

7. A Complete Urban Model

An urban model is a general equilibrium model with housing, land, capital, transportation, labor, and export good markets, plus locational equilibrium conditions. Our analysis so far has been based on incomplete specifications of these markets. This section makes the specifications complete.

7.1. The Housing Market

The demand and supply of housing are given by Equations (17) and (29), respectively. To complete the housing market, we need only specify the market equilibrium condition, namely that demand equals supply. Note that the demand function, Equation (17), applies to a single household, whereas the supply function, Equation (29), applies to all the housing at location *u*. To make the two functions comparable, we must multiply the demand function by $N\{u\}$, the number of households at location u. Thus, the housing market equilibrium condition is:

$$N\{u\}H\{u\} = H_S\{u\} \quad . \tag{43}$$

7.2. The Land Market

The derived demand for land is given by Equation (31). On the supply side, we assume that there is a fixed number of radians of land, ϕ, available for housing at all distances from the city center. This assumption makes it possible to bring into the model land used for transportation or taken up by some geographical features, such as a lake or a harbor, but only in the form of a pie slice. In a city with no transportation or geographical "cut-outs," the available land takes the shape of a circle and ϕ equals 2π, whereas in a city on a large lake, such as Chicago, the available land takes the shape of a semicircle, and ϕ equals π minus land used for transportation.

Because transportation cost depend only on straight-line distance from the CBD, all points u miles from the CBD involve the same transportation cost and hence, are identical as far as households are concerned. The amount of land available u miles from the CBD equals the circumference of the city at that location. The circumference of a circle with a radius of u miles is 2π, and the circumference of a partial circle containing ϕ radians and with a radius of u miles, is ϕu.

Thus, we can write the supply of land as:

$$L\left\{u\right\} = \phi u . \tag{44}$$

Substituting Equation (44) into the demand function, (31), insures equilibrium in the land market.

Another implicit assumption in a standard urban model is that land rent payments disappear from the urban area. This is equivalent to assuming that all land rents are paid to absentee landlords, that is, to people who own the land but live outside the urban area. As indicated on the reading list at the end of this chapter, several scholars have analyzed urban models in which the land rents are paid to people who live in the urban area (and therefore influence their behavior).

7.3. The Capital Market

The assumptions in Section 5 completely specify the capital market. The demand for capital is given by Equation (30), and the supply of capital is given by the assumption that the rental rate of capital, r, is fixed. This assumption is equivalent to the assumption of a horizontal supply curve for capital. In other words, this basic urban model considers capital to be highly mobile across locations.

7.4. The Market for Transportation Services

The market for transportation services is not usually explicitly recognized in the formulation of an urban model. Nevertheless, the assumption that the per-mile cost of commuting, t, is constant requires that enough land be used for transportation that congestion never occurs. Commuting speed, and hence, commuting cost, depends on the capacity of a highway. As the number of people commuting on a highway approaches what is called its design capacity, commuting speed starts to fall and t starts to rise. This curve relating number of commuters to *t*, which is illustrated in Figure 3, can be thought of as the supply curve for transportation services. In effect, therefore, the implicit assumption in a standard model is that the share of land used for transportation is large enough so that even near the CBD where the number of commuters is large, the rising part of this supply curve is never reached.

The implicit transportation market also is important because it gives the model its spatial detail. The assumption that transportation cost is proportional to straight-line distance from the CBD implies that all points a given distance from the CBD are, for the purposes of the model, identical. In effect, this assumption translates the behavioral variable, which is total transportation costs, into a geographical variable, distance from the CBD. Other assumptions about the transportation network, that is, other methods for translating transportation costs into locations, can have strikingly different implications for the spatial arrangement of economic activity.

7.5. Preliminary Treatment of the Labor and Goods Markets

The last two markets to consider are the labor and export good markets, which are closely linked. An urban area cannot exist unless local firms producing an export good, that is, a good sold on a national market, want to hire local residents. In a basic urban model, the export good market is left implicit and the treatment of the labor market is very rudimentary.

On the demand side, export firms, that is, firms exporting products to a national market, are assumed to exist in an urban area's central business district (CBD). These firms hire N workers at the market wage Y. For now, let us assume that N is a variable and Y is a parameter. We will return to these assumptions shortly.

By Assumption 6, all households who live in the urban area have one worker employed in the CBD.[9] As noted earlier, $N\{u\}$ is the number of households (and hence, of workers) living at location u, so the total supply of workers is the integral of this $N\{u\}$ function over all inhabited locations, that is all values of u in the urban area. Assume for simplicity that the CBD does not take up any space. This assumption is convenient, but inessential; the model could easily be solved for a CBD with a fixed radius. Then the limits of integration are from zero to $\bar{u}$, the outer edge of the urban area.

Equilibrium in the labor market requires that demand equal supply, or in symbols,

$$\int_0^{\bar{u}} N\{u\}\, du = N\,. \tag{45}$$

7.6. Locational Equilibrium

The final component of a complete urban model is the spatial dimension. All of the variables in the model are indexed by u, and the pattern of variation across space is driven by the locational equilibrium condition, equation (12), which becomes the bid-price function, Equation (21), when there is a Cobb-Douglas utility function and a constant $T'\{u\}$. The height of the bid-price function is determined by an anchoring equation, such as (38).

7.7. The Complete Model

This complete urban model contains 10 unknowns, $H\{u\}, H_s\{u\}, L\{u\}, K\{u\}, N\{u\},$. $P\{u\}, R\{u\}, N, \bar{u}, and\ U^*$ The first seven of these unknowns are functions, not variables. Although

[9] The model can easily be extended to consider the case of two workers per household, as long as they both work in the CBD.

this fact adds complexity, the model can still be solved with standard algebraic techniques.[10]

The model also contains the following 9 equations (which incorporate the implicit assumptions about the market for transportation services discussed in Section 7.4.):

Housing market: (17), (29), and (43)

Land Market: (31) and (44)

Capital Market: (30)

Labor and Export Good Markets: (45)

Locational Equilibrium: (21) and (38).

Because it has it has 10 unknowns and 9 equations, this model cannot be solved in this form. To find a solution, one additional assumption must be made about the labor and export good markets. We now turn to two alternative versions of this assumption.

8. The Distinction Between Open and Closed Models

A basic urban model can be solved either by assuming that the number of households in an urban area is fixed or by assuming that households are perfectly mobile across urban areas. The former assumption defines a **closed model**, the latter defines an **open model**.

A closed model is appropriate in two cases. First, it is appropriate for analyzing an urban area that makes up an entire nation so that it is difficult for people to move in or out. Singapore is perhaps the best example of this case. Second, a closed model is appropriate for analyzing changes that affect all the cities in a system equally. In this case parameter changes do not alter the relative desirability of each city and no migration between cities occurs. A closed model should be used, for example, to study the impact of a federal gas tax on urban areas.

An open model is appropriate for studying changes that affect one urban area but not others.

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[10] Remember that it is possible to treat $\gamma(or\,\gamma)$ as the tenth variable, instead of U^*. If this approach is taken, one must use Equation (22) as the locational equilibrium condition, instead of Equation (21).

Changes of this type do induce inter-area migration. An open model should be used, for example, to study the impact on urban spatial structure of a local gas tax or of an increase in the value of agricultural land around a single city, or to study differences across the cities in a region.

Two aspects of the distinction between closed and open models need to be emphasized. First, each approach makes exogenous one of the 10 variables listed in the previous section, and thereby results in a manageable system with 9 equations and 9 unknowns. By assuming a fixed population, closed models make population, N, exogenous. We highlight this assumption by writing $\bar{N}$ instead of N for a closed model. Because it assumes that households are mobile across urban areas, an open model requires that identical households achieve the same level of utility regardless of which urban area they live in. As long as one urban area is small relative to the system of urban areas, therefore, the utility level in each urban area can be regarded as fixed at the level in the entire region or nation. Parameter changes in one urban area may induce people to move in or out, so urban population, N, is not fixed, but these changes cannot alter the utility level, which we now write as $\overline{U}^*$, to emphasize that it is fixed both across and within an urban area. In short, the difference between a closed and an open model is that in a closed model N is fixed and U^* is a variable, whereas in an open model U^* is fixed and N is a variable.

The second aspect of the closed-open distinction is their treatment of the labor market. Neither approach specifies the derived demand for labor. As a result, shifts in the supply of labor, which lead to movement along an unspecified demand curve, cannot be analyzed in either model without making the extreme assumption that the derived demand for labor is horizontal, that is, that firms will hire any number of workers at income Y. In contrast, shifts in the demand curve, at least as represented by a change in Y, can be analyzed. In a closed model, the supply of labor, N, is a parameter, so the extreme assumption about demand is only required for analyzing the impact of changes in N on model outcomes. Unfortunately, however, every parameter in an open model except Y affects the supply curve for labor, which is determined endogenously. As a result, comparative static analysis of a standard open urban model cannot be conducted without the above extreme assumption about labor demand. In other words, an extension to a more complete labor market is particularly important for an open model.

These conclusions are illustrated in Figure 4. The dotted line represents an unspecified derived demand curve for labor. The solid vertical line is the supply of labor in a closed model, whereas the upward sloping line is the supply of labor in an open model. The shape of this open-model supply curve will be derived in Section 10. In this figure, a shift in the demand curve can be described by a change in Y and hence, can be analyzed in either model. A shift in either supply curve, however, results in a change in Y, which is not incorporated into either model because Y is treated as a parameter.

9. Solving a Closed Urban Model

The key to solving a closed urban model is to find the rent-distance function. We already completed this step in Sections 5 and 6. The result we need is Equation (40).

The next step is to solve for $N\{u\}$. Equation (43) reveals that $N\{u\}$ is the ratio $H_s\{u\}$ to $H\{u\}$. Moreover, $H\{u\}$ can be expressed in terms of the model's parameters by substituting Equation (41) into (17). Finding $H_s\{u\}$ as a function of these parameters is more complicated. Equation (30) can be solved for $K\{u\}$ as a function of $P\{u\}$ and $H_s\{u\}$. Then (36) can be used to replace $P\{u\}$ with $R\{u\}$. Substituting the resulting expression for $K\{u\}$ into (29), and solving for $H_s\{u\}$ yields:

$$H_s\{u\} = DR\{u\}^{1-a} L\{u\}, \tag{46}$$

where

$$D = A\left(\frac{1-a}{a\,r}\right)^{1-a}. \tag{47}$$

Now we can use Equation (40) to eliminate $R\{u\}$ and Equation (44) to eliminate $L\{u\}$ The result:

$$N\{u\} = \frac{\overline{R}\,\phi\, u\left(Y - tu\right)^{(1/a\alpha)-1}}{a\alpha\left(Y - t\overline{u}\right)^{1/a\alpha}} \quad . \tag{48}$$

The third step is to solve for $\overline{u}$ by carrying out the integration in Equation (45). Substituting (48) into (45), we find that the relevant integral is:

$$\int_0^{\overline{u}} \frac{\overline{R}\,\phi\, u\left(Y - tu\right)^{(1/a\alpha)-1}}{a\alpha\left(Y - t\,\overline{u}\right)^{1/a\alpha}}\, du = \overline{N} \tag{49}$$

To solve this integral, we can make use of the following formula:

$$\int u\left(c_1 + c_2 u\right)^n du = \frac{-1}{\left(c_2\right)^2\left(n+1\right)\left(n+2\right)}\left(c_1 + c_2 u\right)^{n+2} + \frac{u}{c_2\left(n+1\right)}\left(c_1 + c_2 u\right)^{n+1} \tag{50}$$

To apply this formula to Equation (49), we must treat $\overline{R}\phi$ divided by the denominator of (49) as a constant, and set $c_1 = Y, c_2 = -t,$ and $n = \left[(1/\alpha\alpha) - 1\right]$. Thus, the solution to (49) is

$$\int_0^{\overline{u}} N\{u\}\, du = \frac{\overline{R}\,\phi\, b}{\left(Y - tu\right)^{1/a\alpha}}\left[\frac{-\left(Y - tu\right)^{b-1}}{\left(-t\right)^2 b\left(b+1\right)} + \frac{u\left(Y - tu\right)^b}{-tb}\right]_0^{\overline{u}}, \tag{51}$$

where $b = 1/\alpha\alpha$ and the right side must be evaluated at $\overline{u}$ and 0. Completing this evaluation and setting the result equal to $\overline{N}$ yields:

$$\overline{N} = \left(\frac{\overline{R}\,\phi}{t}\right)\left(\frac{Y^{b+1}}{t(b+1)\left(Y - t\overline{u}\right)^{b}} - \frac{Y - t\overline{u}}{t(b+1)} - \overline{u}\right). \tag{52}$$

This equation contains a major disappointment: It cannot be solved explicitly for $\overline{u}$. This fact is an important barrier to analytical manipulation of this model. To obtain a complete solution with particular parameter values, one must employ numerical methods. To be specific, one must program this equation on a computer and increase the value of $\overline{u}$, given the values of the other parameters, until Equation (52) is exactly satisfied.

Finally, we can solve for U^* using Equations (21) and (41). In effect, U^* is a residual in the sense that nothing else in the model depends on it.

10. Solving an Open Urban Model

The equations for an open urban model are exactly the same as the equations for a closed urban model, except that *N* is now treated as a variable and U^* is treated as a constant (and written $\overline{U}^*$).

The derivations presented in Section 7 and 9 carry over to the open model. Thus, Equation (40) gives the bid-rent function, $R\{u\}$, and Equation (48) indicates the population function, $N\{u\}$. With an open model, however, it is possible to use the indirect utility function to solve for $\overline{u}$. Evaluating (21) at location $\overline{u}$ and remembering that $P\{\overline{u}\} = \overline{P} = \overline{R}^{\alpha} / C$ we find that:

$$\overline{U}^* = \frac{k\left(Y - t\overline{u}\right)}{\overline{P}^{\alpha}} = \frac{k\left(Y - t\overline{u}\right)}{\left(\frac{\overline{R}^{a}}{C}\right)^{\alpha}}. \tag{53}$$

Solving (53) for $\overline{u}$ yields:

$$\bar{u} = \frac{Y - \dfrac{\bar{P}^{\alpha}\bar{U}^{*}}{k}}{t} = \frac{Y - \dfrac{\bar{R}^{a\alpha}\bar{U}^{*}}{C^{\alpha}k}}{t} \,. \tag{54}$$

This result has a straightforward interpretation: At a given income and transportation cost, the higher the value of system-wide utility, the smaller must be the commuting distance of the most-distant worker.

With $\bar{u}$ known, we can now use (52) to find N (which now has no bar). Because N is an explicit function of $\bar{u}$, we do not run into the problem that we encountered with a closed model; a complete analytical solution to the open model can easily be obtained.

11. Density Functions and Building Heights

Two important features of an urban area are patterns of population density and building height. Population density is people per square mile, or $N\{u\}/L\{u\}$. From Equations (40), (44), and (48), it follows that

$$D\{u\} = \frac{R\{u\}}{a\alpha(Y - tu)} \tag{55}$$

Using the derivative of (55); the relationship between $R\{u\}$ and $P\{u\}$, Equation (35); and the equation that defines $P\{u\}$, (41), it can be shown that the slope of this density function is always negative. Moreover, this density function is flatter (i.e. the derivative is less negative) than the rent function for all reasonable values of the parameters.

Building height is, roughly speaking, measured by the ratio of capital to land; that is, the more capital used on a given amount of land, the higher buildings are likely to be. To express the capital/land ratio as a function of distance, we begin with Equation (30), substitute in (41) to eliminate $P\{u\}$ and (29) to eliminate $H_S\{u\}$, and solve the resulting equation for $K\{u\}$. The result is

$$K\{u\} = \left(\frac{A(1-a)}{C\, r} \right)^{1/a} R\{u\}\, L\{u\}\,. \tag{56}$$

Dividing both sides by $L\{u\}$, we find that the capital/land function is proportional to $R\{u\}$:

$$\frac{K\{u\}}{L\{u\}} = qR\{u\}\,, \tag{57}$$

where

$$q = \left(\frac{A(1-a)}{C\, r} \right)^{1/a}. \tag{58}$$

This result gives a simple, visual test of the basic urban model: Are buildings tallest in the CBD, with a gradual decline as one moves out toward the suburbs? To a rough approximation, at least, this prediction holds up in many metropolitan areas.

12. Comparative Statistics

An urban model can be used for comparative static analysis, that is, for determining the impact of changes in the parameters on urban spatial structure. This section discusses comparative static analysis of the basic urban model. The key parameters in the basic urban model are $Y, t, \overline{R},$ and $\overline{N}$ (in a closed model) or $\overline{U}$ (in an open model).[11] Comparative static analysis also can be conducted for $\alpha, a, A, r,$ and $\phi.$ The key variables are $P\{u\}, R\{u\}, D\{u\}, \overline{u},$ *and* $\overline{U}$ (in a closed model) or $\overline{N}$ (in an open model).

The strategy for deriving the comparative static results is quite different in open and closed models. Both strategies begin by finding the derivative of $\overline{u}$ with respect to the parameter under consideration, but with an open model, this derivative comes from differentiating Equation (54) and is

[11] An alternative, somewhat more complicated analysis can be conducted using t_o and t_γ from Equation (16) instead of t.

quite simple, whereas with a closed model it comes from differentiating Equation (52) and is very complex. Because $\bar{u}$ appears in most of the other equations of the model, this derivative can then be used to solve for other comparative static derivatives. For example, by totally differentiating Equation (40), which holds in both the open and closed model, we find that the derivative of $R\{u\}$ with respect to a parameter, say δ, can be written:

$$\frac{dR\{u\}}{d\delta} = \frac{\partial R\{u\}}{\partial \delta} + \frac{\partial R\{u\}}{\partial \bar{u}} \frac{d\bar{u}}{d\delta} \quad . \tag{59}$$

The relative simplicity of $d\bar{u}/d\delta$ for all the parameters in the open model implies that this equation, along with other comparative static results, is generally much easier to evaluate in a basic open model than in a basic closed model.

12.1. Illustrative Comparative Static Results for an Open Model

Differentiating Equation (54) with respect to its key parameters yields:

$$\frac{d\bar{u}}{dY} = \frac{1}{t} > 0\,; \quad \frac{d\bar{u}}{dt} = -\frac{\bar{u}}{t} < 0 \tag{60}$$

$$\frac{d\,\bar{u}}{d\,U^*} = -\frac{\bar{P}^{\alpha}}{kt} < 0\,; \quad \frac{d\,\bar{u}}{d\bar{R}} = -\frac{a\alpha U^* \bar{R}^{a\alpha-1}}{k\,t\,C^{\alpha}} < 0\,.$$

Thus, the physical size of an urban area increases with Y, and decreases with $t, U^*, and\ \bar{R}$.

With these results in hand, we can turn to the bid and density functions, as well as the population equation. Totally differentiating Equation (40) with respect $R\{u\}, Y, and\ \bar{u};$ dividing by dY*;* substituting in the above expression for $d\bar{u}/dY;$ and rearranging terms yields:

$$\frac{d\;R\{u\}}{d\;Y} = \frac{R\{u\}}{a\alpha\left(Y - tu\right)} > 0\,. \tag{61}$$

This result implies that an increase in income increases the rent-distance function everywhere. Differentiating the rent-distance function with respect to $\overline{R}$ instead of Y and then following the same procedure (and using Equation (53)) reveals that the derivative of $R\{u\}$ with respect to $\overline{R}$ is zero; in an open model, the rent function is fixed by the fixed-utility assumption and does not shift when $\overline{R}$ changes.

To find the impact of an income change on urban population, we must totally differentiate Equation (52) (with no bar on the N), with respect to $N, Y,$ and $\overline{u}$; divide by dY; substitute in the above expression for $d\overline{u}/dY$; and rearrange terms. The result:

$$\frac{d\ N}{d\ Y} = \frac{\overline{R}\ \phi}{t^2}\left(\frac{Y^b}{\left(Y - tu\right)^b} - 1\right) > 0 \tag{62}$$

Urban population inevitably increases with income.[12] Other comparative static results can be derived in a similar manner.

12.2. Illustrative Comparative Static Results for a Closed Model

Because N is fixed in a closed model, the comparative static results for $\overline{u}$ can be found by differentiating Equation (52). For example, totally differentiating (52) with respect to $\overline{R}$ *and* $\overline{u}$ reveals that

$$\frac{d\overline{u}}{d\overline{R}} = -\frac{t\left(b+1\right)N}{\overline{R}^2\phi b}\frac{1}{\left(\dfrac{Y}{Y - t\overline{u}}\right)^{b+1} - 1} < 0 \tag{63}$$

In words, the physical size of an urban area decreases as agricultural rents increase. This result can be used to pin down other comparative static derivatives. From Equation (40), for example, we find that:

[12] For an derivation and empirical test of this result, see Yinger and Danziger (1978)

$$\frac{d\ R\{u\}}{d\ \bar{R}} = \frac{R\{u\}}{\bar{R}}\left[1 + \frac{tb\bar{R}}{Y - t\bar{u}}\frac{d\bar{u}}{d\bar{R}}\right] \tag{64}$$

By substituting in Equation (63), it can be shown that the term in square brackets, and hence, $dR\{u\}/d\bar{R}$ itself, is positive; in a closed model, unlike an open model, an increase in $\bar{R}$ shifts up the rent function to ensure that there is enough room for the urban area's population. This algebraic derivation is left to the reader (see exercise 7).

Although the algebra is often quite complex, other comparative static results for a closed model can be found in a similar manner.

13. Evidence about Key Urban-Model Results

13.1. Introduction

The basic urban models presented in this chapter make many predictions about the spatial pattern of economic activity. In particular, they predict that the price of housing services (i.e., the price of housing controlling for housing characteristics), the price per unit of land, population density, and building height, all decline with distance from the CBD. Anyone who has looked at a big city skyline or tried to rent an apartment near the center of a big city can verify that these predictions are consistent with casual empiricism. This section reviews some of the more formal tests of urban models.

13.2. Evidence on Housing Prices and Rents

The most basic test of an urban model is to determine whether housing prices and land rents actually do decline with distance from the CBD. As shown by Coulson (1991), a formal statement of this hypothesis is easily derived from a basic household maximization problem. With H held constant, the derivative of apartment rent, $P\{u\}H$, with respect to u is $P'\{u\}H$, which by Equation (11) equals $-T\{u\}$. With a constant commuting cost per mile, this derivative is simply $-t$. Hence, in a regression of apartment rent on distance, controlling for the housing characteristics that determine H, the coefficient of distance is t. By using Equation (4), this approach can also be applied to house values instead of apartment rents, and by using Equation (35), it can be applied to land rents as well.

Many studies, which are reviewed in Coulson (1991) or in Mills and Hamilton (1994), have regressed housing rents or prices on distance from the CBD, controlling for housing characteristics. Virtually all of these studies find that the distance variable is statistically significant. Coulson (1991) carried out such regressions using data for State College, PA, which appears to conform to a relatively homogeneous urban area dominated by a CBD. Coulson finds that in a linear regression the coefficient of u is highly significant statistically, and corresponds to a value of t equal to \$0.527 per mile, or \$0.263 per mile each way.[13] This result corresponds, for example, to gas and oil costs of \$0.10 per mile, a commuting speed of 20 MPH, and commuting time valued at \$3.25 per hour.

Coulson also finds that the linear form can be rejected in favor of a model with a more general functional form, called Box-Cox. As Coulson points out, this result is not a rejection of an urban model, but is instead a rejection of the assumption that the commuting cost per mile is fixed (i.e., that $T\{u\} = tu$). The form for $T\{u\}$ implied by Coulson's results is complex and cannot be determined from the information in his article. Although Coulson's statistical test rejects the linear form, he also finds that the linear model yields results that are qualitatively similar to those using the Box-Cox model, so that assuming constant per-mile transportation costs may be a reasonable approximation.

13.3. Evidence on Density Gradients

Another large empirical literature, which is reviewed in McDonald (1989), focuses on the relationship between population density and distance from the CBD. As shown by Equation (55), a basic urban model predicts that population density will decline with distance from the CBD, so this literature also serves as a test of the basic urban model.

[13] Coulson (1991) presents numbers that are twice this large, but he mistakenly uses a nominal interest rate (set to 10 percent) instead of a real interest rate (which I have set to 5 percent). For more on the importance of using real interest rate in discounting problems, see Yinger et al. (1988). For future reference, it should also be noted that Coulson uses actual street distance, not radial distance, to the CBD and, in some of his models, asks whether travel cost varies with direction. These are valuable extensions of the standard approach, and they appear in some of the studies listed at the end of these notes.

This literature began before urban models were developed when some researchers suspected empirical regularities in the relationship between population density and location. The early studies searched for functional forms that accurately characterized this relationship, and many of them settled on an exponential form, that is a form expressing population density as an exponential function of distance from the city center. Muth (1969) and Mills (1972) later showed that, under some assumptions, this exponential form can be derived from an urban model.

Virtually all studies of population density find that density declines with distance from the city center. This finding provides more general evidence in support of the basic urban model. Moreover, as predicted by an urban model, density functions appear to take similar forms in many different urban areas and to exhibit changes over time that are roughly consistent with comparative static predictions from an urban model.

13.4. Tests Using Functional Forms from Urban Models

The empirical studies reviewed above provide general tests of a basic urban model, but few of them provide very demanding tests of these models, that is, tests that make use of theoretically derived functional forms or theoretically derived restrictions. Exceptions include the Coulson study, which tests (and rejects) the assumption that the per mile cost of commuting is constant, and studies of population density that employ an exponential form, which can be derived from one oversimplified version of an urban model.

A more demanding test of urban models is provided by Yinger (1979), who estimates a price-distance function using the functional forms derived in Section 6. Remember from Equation (4) that $V = P\{u\} H / i$ and remember that H is assumed to be a function of the structural characteristics of housing, say $X_1 \ to \ X_N$. The form of $P\{u\}$ is given by (41). A reasonable form for H is as follows:

$$H = X_1^{\eta_1} X_2^{\eta_2} X_3^{\eta_3} \cdots X_M^{\eta_M} \tag{65}$$

Now combine equations (4), (41), and (65) and take natural logs to obtain:

$$\ln\{V\} = \ln\left\{\frac{\bar{P}}{i}\right\} + \frac{1}{\alpha}\ln\{Y - tu\} - \frac{1}{\alpha}\ln\left\{Y - t\bar{u}\right\} + \sum_{m=1}^{M}\eta_m \ln\{X_m\} \qquad (66)$$

Thus, with data on $V, Y, u, \bar{u}, t,$ and the $X's,$ this equation can be estimated with linear regression techniques. Alternatively, t can be estimated as a nonlinear parameter, which is the procedure followed by Yinger (1979) using data for single-family houses in Madison, WI, and for both apartments and single-family houses in St. Louis.

The results for Madison, which has employment concentrated near its CBD, are very supportive of the model. The estimated value of α, the share of income spent on housing, is very reasonable, 17.2 percent, and is highly significant statistically. Moreover, the estimated value for t is quite reasonable.[14] The results for St. Louis, which has very dispersed employment, are more mixed. At locations close to the CBD, the estimated value of α is about 21 percent for both the rental and sales data. The estimated value of α is not reasonable, however, and indeed sometimes has the wrong sign, more than one mile from the CBD (for the rental data) or two miles from the CBD (for the owner data). In addition, the estimated values of t are not as reasonable as in the Madison equation.

Note that in Equation (66) the coefficients of the two terms containing Y are the same. In fact the functional form used here does not make sense if these two coefficients are different, and the results presented above are based on equations in which they are restricted to be the same. In fact, however, Yinger must reject the hypothesis that these two coefficients are the same. This test suggests that the assumption of a Cobb-Douglas utility function is not correct, although the results based on it are plausible enough, at least when employment actually is concentrated near the CBD, so that it may be a reasonable approximation under some circumstances.

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[14] To be specific, Yinger estimates that t_0 is .05, and *w*/MPH is .025, which corresponds, for example to a w equal to 0.5 and a commuting speed of 20 MPH.

13.5. Conclusions

The most fundamental prediction of an urban model, that housing prices and rents vary with location after controlling for the structural characteristics of housing, is strongly supported by many studies. More detailed predictions from a basic urban model also are supported in urban areas where the assumptions of that model are reasonably accurate. In other words, these predictions are supported in a relatively homogeneous urban area in which employment is relatively centralized. Not surprisingly, the model does not hold up so empirically so well in areas where its assumptions are not met.

Much of urban economics is dedicated to relaxing the assumptions of the basic urban model, that is, to building more general urban models. As indicated earlier, studies that relax each of the assumptions in the basic urban model are listed after this chapter. Empirical work in urban economics also focuses on more complex models; after all, the failure of a basic urban model to accurately predict housing prices or other features of a heterogeneous urban area with diffuse employment, such as St. Louis, cannot be interpreted as a rejection of urban models—only as a rejection of overly simple urban models. The great challenge facing empirical work in urban economics is that formal tests of urban models with general assumptions are difficult to derive. Nevertheless, many scholars have devised clever tests that build on extensions of the logic of a basic urban model to a setting with, for example, multiple worksites or traffic congestion. This empirical literature is obviously way beyond the scope of this chapter.

Exercises

Part I. Bid Functions and Density Functions

1. The slope of the bid-price function, $P\{u\}$, *is* $-t/H$. Use Equation (35) to derive general expression for the slope of the bid-rent function, $R\{u\}$.

2. Derive expressions for the slopes of the bid-rent function, $R\{u\}$, and the population density function, $D\{u\}$, in the case of a Cobb-Douglas utility function.

3. Equation (48) provides an expression for the population function $N\{u\}$ with a Cobb-Douglas utility function. Derive and explain the slope of this function.

4. Derive an expression for the second derivative of the bid-price function, $P\{u\}$, with a Cobb-Douglas utility function.

5. Derive expressions for the second derivatives of the bid-rent function, $R\{u\}$, and the population density function, $D\{u\}$, in the case of a Cobb-Douglas utility function.

Part II. Comparative Statics

6. Derive expressions for the comparative static derivatives $dP\{u\}/d\alpha$ and $dP\{u\}/da$ in a basic open urban model and determine their signs.

7. Prove that the closed-model comparative static derivative $dR\{u\}/d\overline{R}$, which is given by Equation (64), is positive.

8. Derive an expression for the comparative static derivative $d\bar{u}/dY$ in a basic closed urban model. Prove that this derivative is positive. [Hint: One possible approach is to use Equations (40) and (55) to show that if $d\bar{u}/dY$ is negative, $dD\{u\}/dY$ also must be negative, which is a contradiction; if both density and the size of the urban area increase, population cannot remain constant.]

9. Prove that with Cobb-Douglas utility and housing production functions,

(a) in an open urban model, $dR\{u\}/dt < 0$;

(b) in a closed urban model, $dR\{u\}/dt > 0$ for $u < u^*$ and $dR\{u\}/dt < 0$ *for* $u > u^*$, where $0 < u^* < \bar{u}$

[Hint: Use the population integral.]

10. Derive and evaluate the signs of expressions for the following comparative static derivatives in a basic open urban model with Cobb-Douglas utility and housing production functions: $d\bar{u}/dY$, $d\bar{u}/dt$, $d\bar{u}/dU^*$, $d\bar{u}/d\bar{R}$, $dR\{u\}/dY$, $dR\{u\}/d\bar{R}$, $dR\{u\}/dt$, and dN/dY (or $dN\{u\}/dY$).

11. Derive and evaluate the signs of expressions for the following comparative static derivatives in a basic closed urban model with Cobb-Douglas utility and housing production functions: $d\bar{u}/dY$, $d\bar{u}/d\bar{R}$, and $dR\{u\}/d\bar{R}$.

11A. Suppose households in one of the metropolitan areas hit hardest by the foreclosure crisis decide to minimize their risk by cutting back on the share of their income (net of commuting costs) that they devote to housing. This area is called Subprime. Assume this is a long-term effect and that people in other areas do not respond in this way. Use comparative statics analysis to derive the impact of this response on the physical size of Subprime. Keep it simple. Assume Subprime is a monocentric city with one income-taste class and with Cobb-Douglas utility and housing-production functions. [Hint: recall that d/dx of a^x is $a^x\ln(x)$.]

11B. Suppose that, over time, the commuting speeds in Drivealot decline because the popularity of non-work trips goes up, even during rush hour. Use comparative statics analysis (and the simplifying assumptions from part a) to derive the impact of this response on the physical size of Drivealot and on the height of Drivealot's bid function, P{u}.

Part III. Bid Functions with Alternative Utility Functions

12. Derive bid-price and bid-rent functions for a basic urban model with a utility function of the following form:

$$U = a_1 \ln \{Z - \beta H + \omega\} + \alpha_2 \ln \{H\}$$

where the α's, β, *and* ω are constants. Express the bids either as functions of U^*, the fixed utility level, or $\bar{u}$, the outer edge of the urban area. Assume that the housing production function is Cobb-Douglas. For extra credit, derive the population integral.

13. Solve an urban model with a CES utility function. Specifically,

(a) Derive a price-distance function when households have a CES utility function (and all other assumptions for a basic urban model hold). [Hint: Remember that the answer is not altered by a monotonic transformation of the utility function; a CES utility function, unlike a CES production function, does not need the "outer" exponent.]

(b) Derive the corresponding rent-distance function (assuming Cobb-Douglas housing supply).

(c) Derive the population integral for this case (but you do not need to integrate it).

14. Assume that leisure time, *S*, is an argument in a household's utility function, that the utility function is Cobb-Douglas, and that total time available, T, must equal work time, W, plus S plus time spent commuting, which is u / M, where M is commuting speed (miles per hour). In addition, assume that the round-trip operating costs of commuting equal t_0 per mile. Derive a bid-price function based on these assumptions. Compare it to the bid-price function derived in the text.

Part IV. Alternative Urban Models

15. All workers in Bigtwo have identical Cobb-Douglas utility functions and identical skills. Bigtwo has only two employers, each of which is located in the CBD and each of which provides free buses to its employees. Firm A provides fancy buses with a smooth, quiet ride and with many forms of entertainment, such as movies, television, computers, and music. Firm B provides ordinary buses, which are not so smooth and quiet and which provide no entertainment at all.

Workers are perfectly mobile, so they must obtain the same utility level, regardless of where they work. You have observed that employees of both firms live in neighborhoods located exactly $u*$ miles from the CBD.

(a) Which group of workers, employees of firm A or firm B, live outside $u*$?

(b) Which group of workers has higher wages? How much higher?

16. Naderville has a greenbelt, that is, a wooded park, on a ridge that runs in a circle around the CBD. The distance between the CBD and the greenbelt is $u*$ miles. All residents of Naderville have the same incomes, utility functions, and commuting costs. Moreover, they are all very proud of the greenbelt and like to look at it. In fact, residents of Naderville are willing to pay for a better view of the greenbelt from their home. The quality of this view declines with distance from the greenbelt.

More specifically, each resident of Naderville has the following utility function:

$$U = Z^{\alpha_1}\ H^{\alpha_2}\ / u - u^* /^{-\alpha_3}$$

where all the α's are positive and $\alpha 1 + \alpha 2 = 1$.

(a) Derive the price-distance function for Naderville, expressed as a function either of the fixed utility level, $U*$, or of the outer edge of the urban area, $\bar{u}$.

(b) Derive the condition under which the price-distance function in Naderville will be positively sloped for locations inside u^*.

17. Write down a standard urban model with Cobb-Douglas utility and production functions and two income classes. Assume that these two classes have the same utility functions and that commuting costs consist of time costs, which are proportional to income, and on operating costs, which are the same for both groups.

(a) The bid functions of the two groups cross at distance u^* from the CBD. Prove that the higher-income class lives outside u^*.

(b) Derive an expression for the population in each group.

(c) Suppose operating costs are also proportional to income. Does this affect your answer to part (a)? Explain.

18. Derive the so-called land constant for an urban model with a street grid and any pattern of arteries that is not considered in "Around the Block." For example, you could consider vertical and horizontal arteries through the CBD and two vertical arteries that do not go through the CBD.

19. The country of Equalomia is dedicated to equality. The economy of Equalomia is based on free enterprise, subject to various rules designed to make market outcomes more equal. In recognition of the need for work incentives, the government does not insist on complete equality and households in Equalomia fall into one of two income classes, which both contain the same number of people. Households in the richer income class earn exactly twice as much as households in the poorer class. The government has been unwilling to diverge from complete equality in the housing market, however, and every household in Equalomia receives exactly kY^* units of housing services, where k is a constant set by the government and Y^* is average household income. Thus, households can compete with each other for existing apartments, but all apartments contain kY^* units of housing services.

Your job is to describe urban structure in Equalomia. You have discovered that everyone works in the city center; that per-mile commuting costs are constant within an income class; that household's have Cobb-Douglas utility functions; that housing services are produced with a fixed coefficients production function using land and capital (that is, $H_S\{u\} = aL\{u\} = bK\{u\}$, where $H_S\{u\}$ is total housing services supplied at distance u and a and b are constants); and that the amount of residential land is proportional to distance (that is, $L\{u\} = \phi u$, where ϕ is a constant).

20. A bid function, $P\{u\}$ can be derived when amenities, $a\{u\}$, are included in the utility function. Two derivations appear in the literature. The first derivation comes from Polinsky and Shavell. They write an indirect utility function:

$$V^* = V\{P\{u\}, Y - T\{u\}, a\{u\}\},$$

where V^* is a fixed utility level (to reflect household mobility), $V_1 = \partial V / \partial P < 0$, $V_2 = \partial V / \partial (Y - T\{u\}) > 0$, and $a\{u\}$ is defined so that $V_3 = \partial V / \partial a > 0$. Differentiating this function with respect to u, recognizing that $\partial V^* / \partial u = 0$, and solving for $\partial P / \partial u = P'\{u\}$ yields:

$$P'\{u\} = \frac{V_2}{V_1} T'\{u\} - \frac{V_3}{V_1} a'\{u\}.$$

An alternative derivation begins with the following direct utility function:

$$U^* = U\{Z, H, a\{u\}\}.$$

A household's problem is to maximize this utility function with respect to *Z*, *H*, and *u*, subject to

$Y = Z + P\{u\}H + T\{u\}$.

The first-order conditions of this problem imply that

$$P'\{u\} = \frac{\frac{\partial U / \partial a}{\lambda} a'\{u\} - T'\{u\}}{H} = \frac{\frac{\partial U / \partial a}{\partial U / \partial Z} a'\{u\} - T'\{u\}}{H} = \frac{MB_a a'\{u\} - T'\{u\}}{H},$$

where MB_a is the marginal benefit from a unit of a in dollar terms.
Prove that these two expressions for $P'\{u\}$ are equivalent.

21. Use the final equation from question 20 to show that high-income households will live outside low-income households whenever $P'\{u\} < 0$ and

$$\frac{\left(\frac{\partial MB_a}{\partial Y} a'\{u\} - \frac{\partial T\{u\}}{\partial Y}\right) Y}{MB_a a'\{u\} - T'\{u\}} < \frac{\partial H}{\partial Y} \frac{Y}{H}.$$

Interpret this condition.

22. New Orleans has an unusual amenity, namely, elevation. This amenity, labeled *e*, is measured in distance above or below sea level. Elevation is an amenity in New Orleans because of the probability of flooding. New Orleans is protected by levies and pumps. If the levies around the city are breached, the lowest places will flood first, followed by the higher places if the breach and resulting flood are severe enough, particularly if the water overwhelms the pumps. The people of New Orleans don't seem too concerned about this; so far as they are concerned, floods occur so seldom that they are not worth worrying about. Based on past experience, however, home insurance companies care a lot, and they charge more to provide full insurance, including flood coverage, for houses in low-lying neighborhoods. Since flood insurance is required by the government (for the purposes of this problem!), people care about their elevation because it affects the cost of their insurance.

To be more specific, housing, H, is measured in units of (quality-adjusted) square feet, and the price per unit of H is P. The price of a house is $V = PH/r$, where r is the appropriate discount rate. The price of a home insurance policy is set as a percentage of V. This percentage, I, is a function of e. Because the probability of flooding declines with e, so does the price of insurance, so that $I'\{e\} \equiv dI/de < 0$.

Your job is to derive an expression for the price per unit of H, labeled P, as a function of u (= distance from the CBD, which is the only worksite) and e. Use the information in the previous paragraph to specify the household budget constraint. Assume that there is a single household type with a Cobb-Douglas utility function and that all households are homeowners. Make any other simplifying assumptions you want that are appropriate for a basic urban model. Derive and interpret expressions for $P\{u,e\}$ and for $\partial P/\partial e$.

Extra Credit: Suppose you have a sample of houses, with information on values, distance from the city center, elevation, and housing characteristics. You do not know the prices of individual insurance policies, but the insurance industry has told you that the price is a quadratic function of elevation. Derive an estimating equation using the bid function you have derived.

Extra Credit: Insurance companies might set their prices according to the amount of housing capital needed to replace a house, which is more closely related to H than to PH. How would $P\{u,\text{e}\}$ and $\partial P/\partial e$ differ if insurance companies charged per unit of H instead of as a percentage of V?

23. **Greybelt.** Ringcity is part of a large urban area with a beltway. Planners in the region are concerned that the pull of employment to this beltway will result in a "greybelt," defined as a section of the city where there is no residential or commercial

development. Your job is to determine the conditions under which a greybelt will arise, that is, under which there will be a ring in which the bid for land to be used for housing is below the bid for land in non-urban uses.

You may use the standard assumptions for an open urban model with Cobb-Douglas utility and housing production functions. Assume that the beltway is located u^* miles away from the CBD, that it circles the city at this distance, and that the employment along the beltway is constant. Assume that all commuting occurs on radial streets, that there is only one worker per household, that all workers (and households) are identical, and that workers can move between CBD jobs and jobs along the beltway. If you need to make any other assumptions to complete your derivation, just state them as clearly as possible.

Now derive a condition that must be satisfied for a greybelt to arise.