Hedonic Equilibria in Housing Markets: The Case of One-to-One Matching

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Abstract

Neighborhood amenities, including public services, vary across locations, and a well-known theorem says that households with a higher marginal willingness to pay (MWTP) for an amenity sort into locations where the amenity is better. A higher MWTP is equivalent to a steeper bid function for the amenity, and the housing price function, which is the envelope of the household bid functions, reflects this sorting process. This paper derives equilibrium expressions for the amenity level as a function of a household’s relative MWTP under the assumption of one-to-one matching between household classes and amenity levels.

Key Words:  Hedonics, Household Sorting, Neighborhood Amenities  
JEL Codes: H75, R2
Introduction

A famous graph explains the price of a multi-attribute product as a function of one of its attributes in a market with heterogeneous households. Each household has a bid function that indicates how much it would pay for the product at each value of the attribute, and the observed market price function is the envelope of these bid functions. Versions of this graph appear in Rosen’s (1974) seminal article on hedonic markets and in many articles about household sorting across locations with different public services or amenities, including Ellickson (1971), Henderson (1977), Yinger (1982), and Epple et al. (1984). This paper builds on the logic of this graph to derive expressions for the sorting equilibrium with heterogeneous households.

Although the analysis of this graph is similar in the hedonics and local public finance literatures, most public finance and housing studies plot the price per unit of housing services instead of the overall housing price on the vertical axis. Housing services are assumed to be a function of structural housing traits, whereas the price per unit of housing services is a function of local public service levels and neighborhood amenities (both called amenities for conciseness). The implicit prices of structural traits are usually assumed to be constant, which implies that they do not affect household sorting across locations. Thus, the sorting analysis focuses on amenities.

This approach also simplifies the supply side. Rosen models multi-attribute products in general. Heterogeneous firms each have an offer function for each attribute, and the market price function is a joint envelope of the bid and offer functions. Housing suppliers do not usually determine amenities, however, so amenity supply is often assumed to be fixed. This paper treats amenities as fixed, but alternative approaches are briefly discussed in the conclusion.
**Hedonic Equilibria**

A key theorem in local public finance says that households with steeper bid functions, which correspond to a higher marginal willingness to pay (MWTP) for the amenity, win the competition for housing in locations with a higher amenity level. The observed housing price function is the envelope of the underlying household bid functions. See, for example, Epple et al. (1984) or the review in Ross and Yinger (1999). Thus, an equilibrium consists of an allocation of households to locations in which no household has an incentive to move. In other words, this allocation is one in which no household is willing to pay more than the observed price in a location other than its own. The key to understanding this type of equilibrium is to recognize that the slope of the bid-function envelope, which is the market price function, reflects both the decline in MWTP for a single household as the amenity increases and the increase in the MWTP as higher levels of the amenity are allocated to households with steeper bid functions. In this framework, the equilibrium allocation of households to locations with different values of an amenity, which is the focus of this paper, can be described by the relationship between a household’s relative bid-function slope, which is defined more precisely below, and the level of the amenity it receives (that is, the level of the amenity in the location where it makes the winning bid).

These points are shown in Figure 1, which is a version of the famous graph. In panel A, the vertical axis indicates the price per unit of housing services (or its log) and the horizontal axis indicates the level of an amenity, such as a school test passing rate. The thick line is the envelope and the thinner lines are bid functions. Panel B plots the slopes of these curves.² Bid-functions are iso-utility curves, so the slope of an individual bid function declines as the value of the amenity increases. With sorting, however, the slope of the envelope shifts up as sorting replaces
one household type with another that has a steeper bid function at any given level of the amenity. The key here is to distinguish between the actual and relative slopes of the bid functions. Because the slope of a bid function is a household’s MWTP for the amenity at that point, an equivalent distinction is between actual and relative MWTP.

In Figure 1, the envelope, and hence the bid-function of the winning household type, has a smaller actual slope at a passing rate of 90 percent, the highest rate in this example, than at a passing rate of 20 percent. However, the household type that wins the competition for housing at a 90 percent passing rate has a steeper bid-function slope at that passing rate than does any other household type. I define a household’s relative bid-function slope as its slope controlling for the amenity level. Thus, an equivalent statement about this panel is that the household type that wins the competition at a 90 percent passing rate is the type with the highest relative bid-function slope. It is logically possible, of course, that a household with the steepest bid-function at one passing rate does not have the steepest bid function at another passing rate. I follow most of the literature by making the so-called single-crossing assumption, which rules out this possibility. In other words, I assume that bid-function shapes are similar enough across household types so that the type with the steepest bid function at one passing rate also has the steepest bid function at any other passing rate.

In most housing applications, Figure 1 is used to highlight the notion of “capitalization,” which is the impact of an amenity on the price of housing services. This paper shifts the focus to the household sorting process by showing how the level of an amenity a household receives is related to the relative slope of its bid function. This relative slope, unlike the actual slope, reflects only intrinsic differences in amenity demand across households, which is the heterogeneity sorting models are designed to consider, not differences in MWTP that arise
because the sorting process places different households at different points on their amenity demand curves. This relationship highlights an important question about housing markets, namely, how much does relative MWTP have to increase for a household to gain access to locations with better amenities? In addition, when combined with an assumption about the form of household bid functions, this relationship can be used to derive a form for the hedonic price function, that is, for the bid-function envelope (Yinger 2014).

Note that the slope of the hedonic itself cannot be used to shed light on sorting, because it does not distinguish between changes in the slope of the hedonic (as the amenity increases) that reflect movement down a given household’s bid function from changes in slope that reflect a shift from one household type to another. Only the latter changes are relevant for sorting. The appropriate way to describe a sorting equilibrium, therefore, is the relationship between the level of the amenity and relative bid-function slope of the household type that wins the competition for housing at that level.

Further insight into this new focus comes from Rosen’s (1974) well-known two-step empirical procedure. The first step is to estimate the hedonic envelope. The slope of the envelope at a given amenity level is the implicit price of the amenity. The household that wins the competition for housing at that amenity level sets its MWTP equal to this implicit price. In other words, the first-step hedonic reveals one point on each household’s demand function. The second step is to estimate the household demand or inverse demand function for the amenity based on variation in MWTP across households. This second-step regression indicates the impact of demand variables, such as income, on MWTP, holding the amenity level constant. As a result, the income coefficient can be interpreted as the impact of income on relative MWTP. This paper explores the relationship between relative MWTP and amenity outcomes. When
combined with second-step empirical estimates, the results in this paper can be used to determine the impact of a change in demand factors on a household’s access to an amenity.

This paper distinguishes between two types of sorting equilibria, one-to-one and inexact matching, in a housing model with fixed amenities. One-to-one matching is a sorting outcome in which all households with a given relative bid-function slope for an amenity are matched with a unique value for that amenity. As discussed below, many factors influence the slope of a bid function, so the set of households with a given bid-function slope need not be exactly alike; instead, the net impact of their traits on this slope needs to be the same. The new analysis in this paper refers to matching of this form. Inexact matching arises when some household types sort into multiple locations with different values for the amenity (across-location homogeneity) or when multiple household types sort into a single location with one value for the amenity (within-location heterogeneity). As discussed below, inexact matching is addressed by several studies in the sorting literature.

**One-to-One Matching**

To consider one-to-one matching, define $S$ as an amenity, such as public school quality or clean air, and define $\psi$ as an index of a household’s relative bid-function slope. This index is not literally a measure of a bid-function slope, because, as discussed above, this slope may be influenced by the level of $S$ a household receives. Instead, a household $A$ is said to have a higher value of $\psi$ than household $B$ whenever household $A$’s bid function is steeper at a point where the bid functions of the two households cross. As indicated earlier, I also make the “single-crossing” assumption: If household $A$ has a higher value of $\psi$ than household $B$ at one value of $S$, it also has a higher value of $\psi$ at any other value of $S$.

Household heterogeneity is defined by variation in $\psi$. Households with a high relative
MWTP for $S$, as determined by observable factors, such as income, or by unobservable factors, such as education preferences, have a high value for $\psi$. I assume that if two households have the same value for $\psi$ at one value of $S$, they also have the same value of $\psi$ at other values of $S$. In other words, the bid function slope can be written as $\text{MWTP}_i = \psi_i \cdot w\{S\}$, where $w$ is a monotonically increasing function that is common across households.

For example, Yinger (2014) derives bid functions based on multiplicative demand functions for an amenity ($S$) and housing ($H$) in which.

$$\psi_i = \left( \left( K_i^S \exp\{\epsilon_i^S\}\right)^{1/\mu} K_i^H \exp\{\epsilon_i^H\} (Y_i)^{(\theta/\mu) + \gamma} \right)^{-1}$$

where $i$ indicates a household, $K$ indicates a set of observable demand factors, $\epsilon$ indicates a set of unobservable demand factors (including random error), $Y$ is household income, and the superscripts indicate the amenity or housing. The exponents contain the price elasticity of demand for $S$, $\mu$, and the income elasticities of demand for $H$, $\gamma$, and $S$, $\theta$. For ease of presentation, let us refer to a set of households with the same $\psi$ as a household “class,” recognizing that the members of this class may have different demand factors. As in Epple and Platt (1999), for example, a household may be in the same class as another even though it has a lower income because it also has a higher unobserved preference for the amenity. The determinants of $\psi$ may be different, of course, for different amenities.

This paper derives explicit forms for the relationship between $S$ and $\psi$ that can shed light on sorting and be used in sorting models and estimation. As discussed above, household classes with steeper bid functions win the competition for higher levels of the amenity, so the equilibrium can be characterized by the relationship between a household’s relative bid-function slope, $\psi$, and the level of the amenity it obtains, $S$.

With continuous functions, one-to-one matching requires the cumulative distributions of
$S$ and $\psi$ to line up. In other words, for all $\alpha$ between zero and 1, the smallest $(\alpha)(100)$ percent of $S$ values must be linked to the smallest $(\alpha)(100)$ percent of $\psi$ values. Under some circumstances, we can use this definition of one-to-one matching to derive conclusions about the implied relationship between $S$ and $\psi$.

One important category of explicit forms for the equilibrium comes from the following theorem for one-to-one matching with continuous distributions of $S$ and $\psi$:

If the distribution of a monotonically increasing (i.e. order-preserving) transformation of $S$ equals the distribution of a (possibly different) monotonically increasing transformation of $\psi$, then in equilibrium the relationship between $S$ and $\psi$ depends only on the nature of the transformations, not on the post-transformation distribution.

The cumulative distribution associated with this shared distribution is assumed to be strictly increasing.

The proof of this theorem begins with a lemma: With continuous functions, the requirement for a one-to-one match between $S$ and $\psi$, combined with the theorem that households with higher values of $\psi$ receive higher values of $S$, is that the cumulative distributions for $S$, $\phi\{S\}$, and for $\psi$, $\theta\{\psi\}$, always line up. In other words, for any share, $\alpha$,

$$ \int_{S_\alpha}^{S} \phi\{S\}dS = \alpha = \int_{\psi_\alpha}^{\psi} \theta\{\psi\}d\psi \quad \text{(2)} $$

where an “underbar” indicates the lowest observed value of a variable and a “$\alpha$” subscript indicates the value of the variable up to which the cumulative share of observations equals $\alpha$.

These integrals obviously must equal one when $S_\alpha$ equals the highest observed value of $S$ and $\psi_\alpha$ equals the highest observed value of $\alpha$.

Now let $g$ stand for a continuous, monotonically increasing transformation function and define $S^* = g_S\{S\}$ and $\psi^* = g_\psi\{\psi\}$. In addition, let $f$ be the continuous density function for a transformed variable, which, for the theorem considered here, is the same for both $S^*$ and $\psi^*$. The
calculus theorem concerning integration by substitution indicates that for a variable $y$ and a transformation $x = g(y)$,

$$\int_{g(a)}^{g(b)} f(x) dx = \int_{a}^{b} f\{g(y)\}g'(y)dy$$  \hspace{1cm} (3)

In the case of $S^*$ and $\psi^*$, therefore,

$$\int_{g_S\{\Sigma\}}^{g_S\{S^*\}} f\{S^*\}dS^* = \int_{\Sigma}^{S^*} f\{g_S\{S\}\}g'_S\{S\}dS$$  \hspace{1cm} (4)

$$\int_{g_{\psi}\{\psi^*\}}^{g_{\psi}\{\psi\}} f\{\psi^*\}d\psi^* = \int_{\psi}^{\psi^*} f\{g_{\psi}\{\psi\}\}g'_{\psi}\{\psi\}d\psi$$  \hspace{1cm} (5)

Now referring back to equation (2), it is clear that $\phi\{S\} = f\{g_S\{S\}\}g'S\{S\}$ and $\theta\{\psi\} = f\{g_{\psi}\{\psi\}\}g'_{\psi}\{\psi\}$. It follows that one-to-one matching requires

$$\int_{g_S\{\Sigma\}}^{g_S\{S^*\}} f\{S^*\}dS^* = \alpha = \int_{g_{\psi}\{\psi^*\}}^{g_{\psi}\{\psi\}} f\{\psi^*\}d\psi^*.$$  \hspace{1cm} (6)

This equation is equivalent to

$$F\{S^*_\alpha\} = F\{g_S\{S\alpha\}\} = \alpha = F\{\psi^*_\alpha\} = F\{g_{\psi}\{\psi\alpha\}\},$$  \hspace{1cm} (7)

where $F\{x\}$ is the cumulative distribution associated with $f\{x\}$, $S^*_\alpha$ is the level of $S^*$ (and $S_{\alpha}$ is the associated level of $S$) at which $F\{S^*\} = \alpha$, and $\psi^*_\alpha$ is the level of $\psi^*$ (and $\psi_{\alpha}$ is the associated level of $\psi$) at which $F\{\psi^*\} = \alpha$. Applying the inverse of $F$ yields

$$F^{-1}\{F\{S^*_\alpha\}\} = F^{-1}\{F\{g_S\{S\alpha\}\}\} = g_S\{S\alpha\} = F^{-1}\{F\{\psi^*_\alpha\}\} = F^{-1}\{F\{g_{\psi}\{\psi\alpha\}\}\} = g_{\psi}\{\psi\alpha\}.$$  \hspace{1cm} (8)

Because equation (8) holds at all values of $\alpha$, it indicates the value of $S$ that is associated with any given value of $\alpha$ and the value of $\psi$ that is associated with that value of $S$.\textsuperscript{6} (This logical chain obviously could also go from $\alpha$ to $\psi$ to $S$.) In other words, we can drop the subscript on the variables in equation (8) and write

$$g_S\{S\} = g_{\psi}\{\psi\}.$$  \hspace{1cm} (9)
Further simplification is possible because the monotonic \( g \) functions are invertible, yielding \( S \) as a function of \( \psi \) (or vice versa):

\[
S = g_S^{-1}\{g_\psi\{\psi\}\} \equiv \gamma_S\{\psi\} \quad \text{and} \quad \psi = g_\psi^{-1}\{g_S\{S\}\} \equiv \gamma_\psi\{S\}.
\]

(10)

This result proves the theorem: the functional relationship between \( S \) and \( \psi \) depends only on the transformations that make their distributions equal, the \( g_s \), not on the distributions themselves.\(^7\)

If both \( g \) functions are linear, then equation (9) (or (10)) obviously leads to a linear relationship between \( S \) and \( \psi \). It is well known, for example, that the appropriate linear transformation can convert one normal distribution into any other normal distribution. If both \( S \) and \( \psi \) have normal distributions, in other words, the one-to-one matching equilibrium can be characterized by \( S \) as a linear function of \( \psi \) (or vice versa).\(^8\) Several other examples are listed in Panel A of Table 1.

With one-to-one matching, equation (2) can also be used to derive explicit equilibria for a variety of cases in which no transformation gives \( S \) and \( \psi \) the same distribution. Consider a uniform distribution for \( S \) and a linear distribution for \( \psi \) (where the \( \kappa \)s are constants):

\[
\int_{S_\alpha}^{S} k_S dS = k_S (S - S) = \alpha = \int_{\psi}^{\psi} (k_{1\psi} + k_{2\psi}\psi)d\psi = k_{1\psi}\left(\psi_a - \psi\right) + \left(\frac{k_{2\psi}}{2}\right)\left((\psi_a)^2 - (\psi)^2\right)
\]

(11)

or, because this equation holds for any \( \alpha \),

\[
S = \left( S - \left(\frac{k_{1\psi}}{k_S}\right)\psi - \left(\frac{k_{2\psi}}{2k_S}\right)(\psi)^2 \right) + \left(\frac{k_{1\psi}}{k_S}\right)\psi - \left(\frac{k_{2\psi}}{2k_S}\right)(\psi)^2 \equiv c_0 + c_1\psi + c_2(\psi)^2.
\]

(12)

Similar calculations lead to the forms catalogued in Table 1. Many other forms are possible.

In some sorting models or their empirical applications, it may be helpful to characterize the equilibrium using a functional form that is reasonably general but also does not involve too many parameters. One form like this with three parameters (the \( \sigma \)s) is
As indicated in Table 1, several forms are special cases of this general form—or special cases under some circumstances. The entry in the second row and third column of Panel A, for example, is the same as equation (13) with $\sigma_3 = 1/n$. The entry in the second row and first column of panel B corresponds to equation (11), which is obviously a special case of equation (13) when $c_1 = 2\sqrt{c_0 c_2}$. Moreover, even when (13) is not exactly correct, it is a polynomial form that provides a reasonable approximation to many other forms.

When combined with the demand function in Rosen’s second step, equation (13) can be used to show the impact of observable household traits on the sorting equilibrium. More specifically, equation (13) can be used to determine the elasticity of $S$ with respect $\psi$ and the Rosen second step yields the elasticity of $\psi$ with respect to an observable household trait, such as income, $Y$. The inverse form of the Rosen second step is a regression of the implicit price each household faces on the level of the amenity it receives and its demand traits. The household sets its MWTP equal to the implicit price, so the dependent variable in this regression is actual MWTP. Because the regression holds the amenity level, $S$, constant, however, the estimated elasticity of the dependent variable with respect to $Y$ can be interpreted as $(\partial \psi/\partial Y)(Y/\psi)$.

Combining these two elasticities reveals how income (or any other demand trait) influences household sorting.

$$
\frac{\partial S}{\partial Y} \frac{Y}{S} = \left( \frac{\partial S}{\partial \psi} \frac{\psi}{S} \right) \left( \frac{\partial \psi}{\partial Y} \frac{Y}{\psi} \right)
$$

Yinger (2014) provides an example using data from the Cleveland area in 2000. His key measure of school quality ($= S$) is the share of students who enter the twelfth grade and subsequently pass all five state tests (mathematics, reading, writing, science, citizenship). He
estimates that, at the mean values of the variables, \((\partial \psi/\partial Y)(Y/\psi)\) equals 0.623 and that
\((\partial S/\partial \psi)(\psi/S)\) equals 0.628. By equation (14), therefore, \((\partial S/\partial Y)(Y/S)\) equals 0.404. In other words, a one percent increase in income leads to a 0.404 percent increase in school quality by this measure solely because of household sorting.\(^{10}\)

**Inexact Matching**

Many equilibria are not one-to-one, of course. A given household class may live in more than one location with different levels of the amenity and/or many different household classes may live in a location with a given level of an amenity. The path-breaking research by Epple and Sieg (1999) and Epple et al. (2001) accounts for these possibilities by solving and estimating a general equilibrium model of bidding and sorting using community-level data.\(^{11}\) This extension comes at a price, however, as these studies are restricted to a single amenity index and require complicated estimating techniques.

A well-known example of the first possibility, called across-location homogeneity, arises when all households are alike. In this case, the hedonic equilibrium is a bid function, and the envelope concept is not needed. Household heterogeneity appears to be far more extensive than amenity heterogeneity, however. A more realistic possibility is plotted in Figure 2. The plotted envelope assumes one-to-one matching. The steepest bid-function in the figure is tangent to the envelope at a passing rate of 0.28. If no other households have steeper bid functions and another location has a passing rate of 0.3, then, assuming that this picture represents an equilibrium, the household class with this bid function will also win the competition for housing in this other jurisdiction. Above a passing rate of 0.28, therefore, observed prices will follow this household class’s bid function, not the derived bid-function envelope.

Because the envelope and the bid function have similar slopes above this point, the
envelope is likely to provide a reasonable approximation to the bid-function in this case, at least if the locations involved do not exhibit a large range in the amenity. In other words, the assumption of one-to-one matching appears to be a reasonable approximation to cases of across-location heterogeneity so long as a household class, that is, households with the same relative MWTP, does not live in a set of communities with widely differing levels of the amenity.

An important example of the second possibility, within-location heterogeneity, is that of a big city in an analysis of a city-level amenity, such as school-district quality, with heterogeneous households. In their treatment of this issue, Epple and Sieg (1999) and Epple, et al. (2001) do not formally define $\psi$; instead their bid function slopes depend on income and a random taste parameter with an assumed joint distribution. They show that the housing price difference between a big city and the jurisdiction ranked one step higher in service level is determined by the city household class with the steepest bid function, called the marginal household. A higher price in the city would induce the marginal household to leave, and a lower price would attract too many households.

In contrast, a functional form based on one-to-one matching assumes that the housing price difference between these two service levels reflects both bidding and sorting. Because some city household classes have flatter bid functions than the marginal household, the predicted difference in the envelope between these two service levels with sorting understates the difference along the marginal household’s bid function. However, the bid function of the marginal household is tangent to the envelope in the jurisdiction with a somewhat higher amenity level than the city, so the slope of this household’s bid function and the slope of a functional form that assumes one-to-one matching may not diverge very much unless these two jurisdictions have significantly different amenity levels. If the two amenity levels are similar,
this type of clustering limits the variation in $S$ that is available to help estimate the envelope but may not be a significant source of bias in the estimated envelope parameters.\textsuperscript{12}

These points are illustrated in Figure 2. Consider a large city with an amenity level (e.g. school passing rate) of 0.15, the lowest in the area, and another district with the next-lowest rate of 0.2. Then the observed price when the amenity equals 0.15 is on the dotted line, not on the envelope. This dotted line is the marginal household’s bid function; it is tangent to this envelope just below the point where the amenity equals 0.2.

In short, deviations from one-to-one matching result in segments along which observed housing prices follow a bid function, not the envelope. Because bid functions are tangent to the envelope at the edges of these segments, the resulting deviations from the envelope may be small, at least when these segments involve small ranges in the amenity level. Under many circumstances, therefore, a form based on one-to-one matching may provide a reasonable approximation to the actual market price function—even with big cities. More research on the nature of this approximation is needed.

**Conclusions**

Households must compete for housing in locations with high-quality amenities. The outcome of this competition can be described by the relationship between the amenity level the household receives and its relative marginal willingness to pay for the amenity. This paper shows that this equilibrium relationship reflects the distributions of the amenity and of households’ relative MWTP and derives expressions for this equilibrium under various assumptions about these distributions—in the case of one-to-one matching. One key result is that if a monotonic transformation of the amenity has the same distribution as a monotonic transformation of relative bid-function slopes, then the equilibrium relationship between the amenity and the slope depends
only on these transformations, not on the post-transformation distribution. Linear transformations, for example, lead to a linear equilibrium relationship between these two variables.

The analysis in this paper assumes that housing markets can be characterized by one-to-one matching, defined as an equilibrium in which every household class, defined by its relative MWTP, receives a unique value of the amenity. One-to-one matching models use continuous mathematics to approximate the allocation of households to locations. These models are obviously not appropriate for analyzing household allocation for amenities with little variation in an urban area, but they appear to provide reasonable approximations under many other circumstances. A key issue for future research on sorting is to provide a more complete comparison of models based on one-to-one matching, which require strong assumptions about matching but do not require strong assumptions about amenity demand or household heterogeneity, and models based on in-exact matching, which provide a more general treatment of matching but currently require a single amenity index and a specific parameterization of household heterogeneity.\(^{13}\)

The assumption that amenities are fixed makes sense in many cases, particularly in the short-run. As many scholars have pointed out, however, the households who sort into a given jurisdiction may change the levels of public services offered there. Epple et al. (2001) include this possibility in the general equilibrium model they estimate. Perhaps future research can show how this possibility alters an equation linking relative bid-function slopes and amenity outcomes. Another issue for future research is that household heterogeneity may lead toward one-to-one matching for some amenities, as each household class manipulates the political or economic environment to generate its own unique amenity level.
Table 1. Selected Forms for a Sorting Equilibrium

Panel A: Same Continuous Distribution for $S^*$ and $\psi^*$

<table>
<thead>
<tr>
<th>Form for $g_{\psi}{\psi}$</th>
<th>Form for $g_S{S}$</th>
<th>$n$-th degree power function$^a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_{\psi}{\psi} = \psi$</td>
<td>Equality$^b$</td>
<td>Linear$^b$</td>
</tr>
<tr>
<td>Linear</td>
<td>Linear$^b$</td>
<td>$1/n$-th degree power function$^b$</td>
</tr>
<tr>
<td>$m$-th degree power function$^a$</td>
<td>$m$-th degree power function$^b$</td>
<td>Constrained polynomial</td>
</tr>
</tbody>
</table>

Panel B: Different Continuous Distributions for $S$ and $\psi$

<table>
<thead>
<tr>
<th>Distribution for $\psi$</th>
<th>Distribution for $S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform</td>
<td>Uniform</td>
</tr>
<tr>
<td>Linear</td>
<td>Linear$^b$</td>
</tr>
<tr>
<td>Quadratic</td>
<td>Quadratic$^c$</td>
</tr>
<tr>
<td>$n$-th degree polynomial</td>
<td>$(n+1)$-th degree polynomial$^c$</td>
</tr>
<tr>
<td>Inverse</td>
<td>Logarithmic</td>
</tr>
<tr>
<td>Exponential</td>
<td>Exponential</td>
</tr>
<tr>
<td>Sine</td>
<td>Cosine</td>
</tr>
<tr>
<td>$m$-th degree power function$^a$</td>
<td>$m$-th degree power function$^b$</td>
</tr>
<tr>
<td></td>
<td>$(m/n)$-th degree power function$^{b,d}$</td>
</tr>
</tbody>
</table>

Note: This table provides examples of sorting equilibrium in which $S$ can be expressed as an explicit function of $\psi$; reversing the rows and columns would yield examples in which $\psi$ can be expressed as an explicit function of $S$. Blank cells indicate complex forms.

$^a$ An $n$-th degree power function is defined as $y = cx^n$.

$^b$ This form is a special case of equation (13).

$^c$ This form is special case of equation (13) if certain restrictions hold; otherwise, the form is approximated by this equation.

$^d$ This cases assumes that the lower bounds of $S$ and $\psi$ equal zero or else that the distributions apply to linear transformations of $S$ and $\psi$ that start at the origin.
At any given level of $S$, $\psi(S)$ is the increase in the bid-function slope due to sorting (depicted as a vertical shift).
Figure 2. Inexact Sorting
*The author is a professor of economics and public administration at the Maxwell School, Syracuse University. I am grateful to Phuong Nguyen-Hoang and Jan Ondrich for helpful comments.

References


Endnotes

1 For literature reviews, see Taylor (2008) on hedonics and Ross and Yinger (1999) on local public finance. Tiebout (1956) is a key paper in the local public finance literature; Rosen (1974) recognized that his model was “similar in spirit to Tiebout’s (1956) analysis of the implicit market for neighborhoods, local public goods being the ‘characteristics’ in this case” (p. 40).

2 A figure comparable to Panel B appears in Rosen (1974, Fig. 4)—but without a decomposition of the bidding and sorting components of the slope.

3 School capitalization studies are reviewed in Nguyen-Hoang and Yinger (2011).

4 As is well known, the implicit price is endogenous in this second-regression regression. See Taylor (2008) for a review of methods that have been developed to address this endogeneity.

5 The associated pre-property-tax bid function is $\hat{P}(S) = C + \psi S^{(1+\mu)/\mu}$, where $\psi$ is the price elasticity of demand for $H(S)$, $C$ is a constant, and parentheses in an exponent indicate the Box-Cox form. See Yinger (2014).

6 Imagine this two-quadrant diagram: $\psi$ is measured upward from the origin on the $y$-axis, and $\alpha$ is measured downward. $S$ is measured on the $x$-axis. The bottom quadrant plots $S_\alpha\{\alpha\}$, which obviously depends on the distribution of $S$, and the top quadrant plots $\psi = g_\psi^{-1}\{g_S\{S\}\}$. Starting from any value of $\alpha$, one can first find the associated $S$ in the bottom quadrant and then find $\psi$ in the top quadrant.

7 Equation (10) also proves that the theorem could be stated with a single transformation (for either $S$ or $\psi$) instead of with two transformations. The symmetric version is used for clarity.

8 An equivalent statement is that if both $S$ and $\psi$ have normal distributions, then separate linear transformations can turn each distribution into a standard normal. In this linear case, if $S$ and $\psi$
have truncated normal distributions, then so do \( g_S(S) \) and \( g_\psi(\psi) \)—so long as the truncation boundaries are the same.

9 Yinger’s second step regresses \( \psi \), not the implicit price, on demand traits. This approach also yields \( \frac{\partial \psi}{\partial Y} \), but it avoids the endogeneity problem in the Rosen second step.

10 In the Cleveland data, the mean owner income is \$38,136\) and the standard deviation, SD, is \$15,762. Starting from the mean, a one-SD increase equals an increase of 41.33 percent. The elasticity in the text indicates that this increase will lead to a \( (0.404)(41.33) = 16.70 \) percent increase in school quality, \( S \), due to sorting; \( S \) has a mean of 32.0 and a SD of 20.4, so a 16.7 percent increase is \( (0.167)(32) = 5.34 \), which is about one-quarter of a standard deviation.

11 This empirical approach builds on the earlier conceptual work by Epple et al. (1984) and Epple and Platt (1998). Moreover, it has been incorporated into model of housing sales prices by Epple, Perez, and Sieg (2010).

12 Even in this case, however, observations within the city cannot be used in a second-step regression because variation in \( \psi \) cannot be observed.

13 As shown by Yinger (2014), a model based on one-to-one matching also leads to a much simpler estimating procedure for the hedonic regression than does a model with inexact sorting.