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Hedonic Equilibria in Housing Markets: The Case of One-to-One Matching

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Abstract

Neighborhood amenities, including public services, vary across locations, and a well-known theorem says that households with a higher marginal willingness to pay (MWTP) for an amenity sort into locations where the amenity is better. A higher MWTP is equivalent to a steeper bid function for the amenity, and the housing price function, which is the envelope of the household bid functions, reflects this sorting process. This paper derives equilibrium expressions for the amenity level as a function of a household's relative MWTP under the assumption of one-to-one matching between household classes and amenity levels.

Key Words: Hedonics, Household Sorting, Neighborhood Amenities
JEL Codes: H75, R2

Introduction

One of the central problems facing local public finance and urban economics is how to derive a housing market equilibrium in which heterogeneous households are sorted across diverse locations. A broad consensus concerning the best way to model this equilibrium has emerged in the literature (see Ross and Yinger 1999), but specific forms for this equilibrium are difficult to obtain. This paper derives expressions for this sorting equilibrium in the special case of one-to-one matching, which is defined as an equilibrium in which each household type is matched with a unique value of a public service or neighborhood amenity.

The consensus model of this equilibrium can be described by a famous bidding-sorting graph in which the price of a multi-attribute product is expressed as a function of one of its attributes. Each household type has a bid function that indicates how much it would pay for the product at each value of the attribute, holding other attributes constant, and the observed market price function is the envelope of these bid functions. Versions of this graph appear in Rosen's (1974) seminal article on hedonic markets and in many articles about household sorting across locations with different public services or amenities, including Ellickson (1971), Henderson (1977), Yinger (1982), and Epple et al. (1984).[1] This paper builds on the logic of this graph.

Although the analysis of this graph is similar in the hedonics and local public finance literatures, many public finance and housing studies plot the price per unit of housing services instead of the overall housing price on the vertical axis. Housing services are assumed to be a function of structural housing traits, whereas the price per unit of housing services is a function of local public service levels and neighborhood amenities (both called amenities here for conciseness). A typical approach is to assume that the function linking structural traits with housing services is common across households and that households can adjust the quantity of housing services; in this case, structural traits do not affect household sorting across locations.[2] Thus, the sorting analysis focuses on amenities. I follow this strategy here.

This approach also simplifies the supply side. Rosen models multi-attribute products in general. Heterogeneous firms each have an offer function for each attribute, and the market price function for an attribute is a joint envelope of the bid and offer functions. Housing suppliers do not usually determine neighborhood amenities, however, so amenity supply is usually assumed to be fixed. This paper also makes this assumption.[3] With this approach, a given amount of housing is available at each amenity level, and households must compete for entry into the most desirable neighborhoods.

Hedonic Equilibria

A key theorem in local public finance, which is consistent with the bidding-sorting graph, says that households with steeper bid functions, that is, with a higher marginal willingness to pay for the amenity per unit of housing services (MWTP), win the competition for housing in locations with a higher amenity level. The observed price function for housing services is the envelope of the underlying household bid functions. An equilibrium consists of an allocation of households to locations such that no household has an incentive to move. In this allocation, in other words, no household is willing to pay more than the observed price in a location other than its own.

The key to understanding this type of equilibrium is to recognize that the slope of the bid-function envelope, which is the market price function, reflects both the decline in MWTP for a single household as the amenity increases and the increase in the MWTP as higher levels of the amenity are allocated to households with steeper bid functions. In this framework, the equilibrium allocation of households to locations with different values of an amenity, which is the focus of this paper, can be described by the relationship between a household's relative bid-function slope, which is defined more precisely below, and the level of the amenity it receives (that is, the level of the amenity in the location where it makes the winning bid).

These points are shown in Figure 1, which is a version of the bidding-sorting graph.[4] In panel A, the vertical axis indicates the price per unit of housing services (or its log) and the horizontal axis indicates the level of an amenity, such as a school test passing rate. The thick line is the envelope and the thinner lines are bid functions. Panel B plots the slopes of these curves. Bid-functions are iso-utility curves, so the slope of an individual bid function declines as the level of the amenity increases. With sorting, however, the slope of the envelope shifts up as sorting replaces one household type with another that has a steeper bid function at any given level of the amenity. The key here is to distinguish between the actual and relative slopes of the bid functions. Because the slope of a bid function is a household's MWTP for the amenity at that point, an equivalent distinction is between actual and relative MWTP.

In Figure 1, the envelope, and hence the bid-function of the winning household type, has a smaller actual slope at a passing rate of 90 percent, the highest rate in this example, than at a passing rate of 20 percent. However, the household type that wins the competition for housing at a 90 percent passing rate has a steeper bid-function slope *at that passing rate* than does any other household type. I define a household's *relative* bid-function slope as its slope controlling for the amenity level. Thus, an equivalent statement about this panel is that the household type that wins the competition at a 90 percent passing rate is the type with the highest relative bid-function slope. It is logically possible, of course, that a household with the steepest bid-function at one passing rate does not have the steepest bid function at another passing rate. I follow the literature by making the so-called single-crossing assumption, which rules out this possibility. In other words, I assume that bid-function shapes are similar enough across household types so that the type with the steepest bid function at one passing rate also has the steepest bid function at any other passing rate.

In most housing applications, Figure 1 is used to highlight the notion of "capitalization," which is the impact of an amenity on the price of housing services.[5] This paper shifts the focus to the household sorting process by showing how the level of an amenity a household receives is related to the relative slope of its bid function. This relative slope, unlike the actual slope, reflects only intrinsic differences in amenity demand across households, which is the heterogeneity sorting models are designed to consider, not differences in MWTP that arise because the sorting process places different households at different points on their amenity demand curves. This relationship highlights an important question about housing markets, namely, how much does relative MWTP, or its determinants such as income, have to increase for a household to gain access to locations with better amenities? In addition, when combined with an assumption about the form of household bid functions, this relationship can be used to derive a form for the hedonic price function, that is, for the bid-function envelope (Yinger 2015).

Note that the slope of the hedonic itself cannot be used to shed light on sorting, because it does not distinguish between changes in the slope of the hedonic (as the amenity increases) that reflect movement down a given household's bid function from changes in slope that reflect a shift from one household type to another. Only the latter changes are relevant for sorting. The appropriate way to describe a sorting equilibrium, therefore, is the relationship between the level of the amenity and relative bid-function slope of the household type that wins the competition for housing at that level.

Further insight into this new focus comes from Rosen's (1974) well-known two-step empirical procedure. The first step is to estimate the hedonic envelope. The slope of the envelope at a given amenity level is the implicit price of the amenity. The household that wins the competition for housing at that amenity level sets its MWTP equal to this implicit price. In other words, the first-step hedonic reveals one point on each household's demand function. The second step is to estimate the household demand or inverse demand function for the amenity based on variation in MWTP across households.[6] This second-step regression indicates the impact of demand variables, such as income, on MWTP, holding the amenity level constant. As a result, the income coefficient can be interpreted as the impact of income on relative MWTP. This paper explores the relationship between relative MWTP and amenity outcomes. When combined with second-step empirical estimates, the results in this paper can be used to determine the impact of a change in demand factors on a household's access to an amenity.

Epple and Sieg (1999), Epple, Romer and Sieg (2001), and Epple, Peress, and Sieg (2010) bring equilibrium notions directly into an empirical procedure.[7] Following the earlier literature, especially Epple, Filimon, and Romer (1984) and Epple and Platt (1998), these studies estimate a general equilibrium model in which the allocation of households across communities meets the conditions of "(a) *boundary indifference*: households on the 'boundary' between two adjacent communities are indifferent between the two communities; (b) *stratification*: the distribution of households across communities exhibits stratification by income and tastes; and (c) *ascending bundles*: the levels of public good provision, housing prices, and the highest income for each type α all ascend in the same order" (Epple, Romer, and Sieg 2001, pp. 1441-2). This approach can account for cases in which people with different preferences live in the same location or in which people with given preferences live in different locations.

This paper does not expand the theory of household sorting. Instead, it derives a new way to characterize the sorting equilibrium in the important limiting case of continuous amenities and one-to-one matching. One-to-one matching is a sorting outcome in which all households with a given relative bid-function slope for an amenity are matched with a unique value for that amenity. As discussed below, many factors influence the slope of a bid function, so the set of households with a given relative bid-function slope need not be exactly alike; instead, the net impact of their traits on this slope needs to be the same. The new analysis in this paper shows that under many circumstances the sorting equilibrium with continuous amenities and one-to-one matching can be expressed in a simple way. Moreover, this approach can be extended to consider multiple amenities.

A more general treatment of matching allows for the possibility that some household types sort into multiple locations with different values for the amenity (across-location homogeneity) and the possibility that multiple household types sort into a single location with one value for the amenity (within-location heterogeneity). Simple descriptions of the sorting equilibrium are not available in these cases, to which we return in a later section.

One-to-One Matching

To consider one-to-one matching, define S as a continuous amenity, such as public school quality or clean air, and define ψ as an index of a household's relative bid-function slope. This index is not literally a measure of a bid-function slope, because, as discussed above, this slope may be influenced by the level of S a household receives. Instead, household A is said to have a higher value of ψ than household B whenever household A's bid function is steeper at a point where the bid functions of the two households cross. As indicated earlier, I also make the "single-crossing" assumption: If household A has a higher value of ψ than household B at one value of S, it also has a higher value of ψ at any other value of S.

Household heterogeneity is defined by variation in ψ. Households with a high relative MWTP for S, as determined by observable factors, such as income, or by unobservable factors, such as education preferences, have a high value for ψ. I assume that if two households have the same value for ψ at one value of S, they also have the same value of ψ at other values of S. In other words, the bid function slope can be written as $\text{MWTP}_i = \psi_i w\{S\}$, where w is a monotonically increasing function that is common across households.

For example, Yinger (2015) derives household bid functions based on multiplicative demand functions for an amenity (S) and housing (H) in which.

$$\psi_i = \left(\left(K_i^S \exp\{\varepsilon_i^S\} \right)^{1/\mu} K_i^H \exp\{\varepsilon_i^H\} \left(Y_i \right)^{(\theta/\mu)+\gamma} \right)^{-1} \tag{1}$$

where i indicates a household, K indicates a set of observable demand factors, ε indicates a set of unobservable demand factors (including random error), Y is household income, and the superscripts indicate the amenity or housing.[8] The exponents contain the price elasticity of demand for S, μ, and the income elasticities of demand for H, γ, and S, θ. For ease of presentation, let us refer to a set of households with the same ψ as a household "class," recognizing that the members of this class may have different demand factors. As in Epple and Platt (1998), for example, a household may be in the same class as another even though it has a lower income because it also has a higher unobserved preference for the amenity. The determinants of ψ may be different, of course, for different amenities.

This paper derives explicit forms for the relationship between S and ψ that can shed light on sorting and be used in sorting models and estimation. As discussed above, household classes with steeper bid functions win the competition for higher levels of the amenity, so the equilibrium can be characterized by the relationship between a household's relative bid-function slope, ψ, and the level of the amenity it obtains, S.

A Class of Equilibria with One-to-One Matching

With continuous functions, one-to-one matching requires the cumulative distributions of *S* and *ψ* to line up. In other words, for all *α* between zero and 1, the smallest (*α*)(100) percent of *S* values must be linked to the smallest (*α*)(100) percent of *ψ* values. Under some circumstances, we can use this definition of one-to-one matching to derive conclusions about the implied relationship between *S* and *ψ*.

One important category of explicit forms for the equilibrium comes from the following theorem for one-to-one matching with continuous distributions of *S* and *ψ*:

> If the distribution of a monotonically increasing (i.e. order-preserving) transformation of *S* equals the distribution of a (possibly different) monotonically increasing transformation of *ψ*, then in equilibrium the relationship between *S* and *ψ* depends only on the nature of the transformations, not on the post-transformation distribution.

The cumulative distribution associated with this shared distribution, i.e. the common distribution into which the distributions of *S* and *ψ* are transformed, is also assumed to be strictly increasing.

The proof of this theorem begins with a lemma: With continuous functions, the requirement for a one-to-one match between two variables, *X* and *Y*, combined with the assumption that higher values *X* are associated with higher values of *Y*, is that the cumulative distributions for *X* and *Y* always line up. Let ϕ indicate the distribution of *X* and *θ* indicate the distribution of *Y*. Then for any share, *α*,

$$\int_{\underline{X}}^{X_\alpha} \phi\{X\}dX = \alpha = \int_{\underline{Y}}^{Y_\alpha} \theta\{Y\}dY \ , \qquad (2)$$

where an “underbar” indicates the minimum value of a variable and an “*α*” subscript indicates the value of the variable up to which the cumulative share of observations equals *α*. These integrals obviously must equal one when X_α equals the maximum value of *X* and Y_α equals the maximum value of *Y*.

Now consider the relationship between S and ψ with one-to-one matching. Let g stand for a continuous, strictly increasing transformation function and define $S^* = g_S\{S\}$ and $\psi^* = g_\psi\{\psi\}$. In addition, let f be the continuous density function for a transformed variable, which, for the theorem considered here, is defined to be the same for both S^* and ψ^*. The calculus theorem concerning integration by substitution indicates that for a variable y and a transformation $x = g\{y\}$,

$$\int_{g\{a\}}^{g\{b\}} f\{x\}dx = \int_a^b f\{g\{y\}\}g'\{y\}dy \tag{3}$$

In the case of S^* and ψ^*, therefore,

$$\int_{g_S\{\underline{S}\}}^{g_S\{S_\alpha\}} f\{S^*\}dS^* = \int_{\underline{S}}^{S_\alpha} f\{g_S\{S\}\}g'_S\{S\}dS \tag{4}$$

$$\int_{g_\psi\{\underline{\psi}\}}^{g_\psi\{\psi_\alpha\}} f\{\psi^*\}d\psi^* = \int_{\underline{\psi}}^{\psi_\alpha} f\{g_\psi\{\psi\}\}g'_\psi\{\psi\}d\psi \tag{5}$$

Now referring back to the lemma in equation (2), $f\{g_S\{S\}\}g'_S\{S\}\}$ is analogous to ϕ and $f\{g_\psi\{\psi\}\}g'_\psi\{\psi\}\}$ is analogous to θ. It follows that one-to-one matching requires

$$\int_{g_S\{\underline{S}\}}^{g_S\{S_\alpha\}} f\{S^*\}dS^* = \alpha = \int_{g_\psi\{\underline{\psi}\}}^{g_\psi\{\psi_\alpha\}} f\{\psi^*\}d\psi^* \ . \tag{6}$$

This equation is equivalent to

$$F\{S_\alpha^*\} = F\{g_S\{S_\alpha\}\} = \alpha = F\{\psi_\alpha^*\} = F\{g_\psi\{\psi_\alpha\}\} \ , \tag{7}$$

where $F\{x\}$ is the cumulative distribution associated with $f\{x\}$, S_α^* is the level of S^* (and S_α is the associated level of S) at which $F\{S^*\} = \alpha$, and ψ_α^* is the level of ψ^* (and ψ_α is the associated level of ψ) at which $F\{\psi^*\} = \alpha$. Applying the inverse of F yields

$$\begin{aligned} &F^{-1}\{F\{S_\alpha^*\}\} = F^{-1}\{F\{g_S\{S_\alpha\}\}\} = S_\alpha^* = g_S\{S_\alpha\} \\ &= F^{-1}\{F\{\psi_\alpha^*\}\} = F^{-1}\{F\{g_\psi\{\psi_\alpha\}\}\} = \psi_\alpha^* = g_\psi\{\psi_\alpha\} \ . \end{aligned} \tag{8}$$

Because S^* and ψ^* have the same distribution, equation (8) indicates that $S^*_\alpha = \psi^*_\alpha$ for all values of α, so we can drop the subscript on the variables and write[9]

$$g_S\{S\} = g_\psi\{\psi\} . \tag{9}$$

Further simplification is possible because the strictly increasing *g* functions are invertible, yielding *S* as a function of ψ (or vice versa):

$$S = g_S^{-1}\{g_\psi\{\psi\}\} \equiv \gamma_S\{\psi\} \text{ and } \psi = g_\psi^{-1}\{g_S\{S\}\} \equiv \gamma_\psi\{S\}. \tag{10}$$

This result proves the theorem: the functional relationship between *S* and ψ depends only on the transformations that make their distributions equal, the *g*s, not on the distributions themselves. Equation (10) also proves that the theorem could be stated with a single transformation (for either *S* or ψ) instead of with two transformations.

If both *g* functions are linear, for example, then equation (9) (or (10)) obviously leads to a linear relationship between *S* and ψ. It is also well known that the appropriate linear transformation can convert one normal distribution into any other normal distribution. If both *S* and ψ have normal distributions, in other words, the one-to-one matching equilibrium can be characterized by *S* as a linear function of ψ (or vice versa).[10]

Finally, note that this derivation applies to a single amenity, but can be extended to multiple amenities under some circumstances. The *i*th amenity has a distribution of values (S_i) and its own demand function, which leads to a distribution of ψ_i . So long as the demand for S_i does not depend on S_j, this analysis can be extended to any number of amenities, each treated independently.[11] As discussed below, however, empirical studies of sorting may run into difficulty if amenities are highly correlated with each other.

In some sorting models or their empirical applications, it is helpful to characterize the equilibrium using a functional form that is reasonably general but also does not involve too many parameters. One form like this with three parameters (the σs) is

$$S = \left(\sigma_1 + \sigma_2 \psi\right)^{\sigma_3} \quad . \tag{11}$$

With $\sigma_3 > 0$, the standard sorting theorem predicts that $\sigma_2 > 0$, too; that is, this theorem predicts that, in equilibrium, steeper relative bid-function slopes, ψ, will be associated with higher values of the amenity, S.

Tables 1 to 3 demonstrate that equation (11) is consistent with a wide range of assumptions about the transformations that lead to the same distribution for S and ψ or about the distributions of S and ψ before transformation.

Table 1 considers the equilibria generated by some simple transformations of S and ψ, under the assumption that the post-transformation distributions of these two variables are the same. Many of these cases lead to equilibria that can be described by equation (11). If the transformations of S and ψ are both power functions, for example, the equilibrium can be characterized by equation (11) with $\sigma_1 = 0$, $\sigma_2 > 0$, and $\sigma_3 > 0$. If both transformations are linear, or if S and ψ have the same distribution after a linear transformation of only one of them, then the equilibrium can be characterized by equation (11) with $\sigma_3 = 1$. The cases marked "N.A." also have well-defined sorting equilibria, but these equilibria do not fit the form of (11).

Table 2 lists some cases in which the distributions of S and ψ take the same form but have different parameter values. The first panel covers proportional transformations that yield sorting equilibria in which $\sigma_1 = 0$, $\sigma_2 > 0$, and $\sigma_3 = 1$. The second panel covers linear transformations. If both S and ψ have Raleigh distributions, for example, equation (11) with $\sigma_3 =$ 1 provides a description of the sorting equilibrium. This panel shows that the same result applies to a variety of other distributions, as well. The third panel indicates that if the distributions of S and ψ both take the log-normal form, the sorting equilibrium can be characterized by (11) with σ_2

=1, and σ_3 >0.

Finally, Table 3 provides examples of distribution pairs for S and ψ that lead to sorting equilibria consistent with equation (11). Some amenities are expressed as proportions, for example, and may follow a standard uniform distribution. If, in addition, ψ follows a beta distribution with parameters 1/a and 1, then the sorting equilibrium fits (11) with $\sigma_1 = 0$, $\sigma_2 = 1$, and $\sigma_3 = 1/a$. Another example is that a standard exponential distribution for S and a Weibull distribution for ψ lead to a sorting equilibrium in which $\sigma_1 = 0$ and the other two parameters of (11) take on positive values.[12]

In short, the sorting equilibrium described by equation (11) is consistent with a wide variety of assumptions about the distributions of S and ψ. Moreover, (11) is also a polynomial form that may provide a reasonable approximation to the equilibria implied by distributions that are not covered by Tables 1 through 3.

Empirical Applications

These are theoretical derivations. In practice, of course, we observe samples, not density functions. Even if we are confident that, after the appropriate transformations, ψ and S have the same continuous distribution, we still must recognize that these two transformed variables are drawn separately from this distribution and that the observed values will therefore not be identical. Because of sorting and one-to-one matching, this situation can be characterized with order statistics. In this type of equilibrium, the lowest value of S^* is matched with the lowest value of ψ^*, the second-lowest value of S^* is matched with the second-lowest value of ψ^*, and so on. In other words, the first order statistic of S^*, $S^*_{(1)}$, is matched with $\psi^*_{(1)}$, and on through the highest order statistics, $S^*_{(n)}$ and $\psi^*_{(n)}$, where n is the sample size, i.e. the number of amenity levels and household types.

Suppose we draw a sample of size n from the probability density $f(x)$, with cumulative distribution function $F(x)$. Then it is possible to derive the asymptotic distribution for the associated (np)th order statistic, where $0 < p < 1$.[13] To be specific, this order statistic "is asymptotically distributed as a normal distribution with mean ξ_p and variance $p(1\text{-}p)/(n[\mathrm{f}(\xi_p)]^2)$," where "$\xi_p$ is the unique solution in x of $F(x) = p$" (Mood, Graybill, and Boes 1974, p, 257). This result implies that the (np)th order statistic equals ξ_p plus a normal error that goes to zero as n gets large. Because S^* and ψ^* have the same distribution, they have the same distribution for their (np)th order statistic; that is, these two order statistics are independent draws from the same order-statistic distribution. Moreover, the difference between these two order statistics is the difference between their two normal errors; this difference is also normal with mean zero and a variance equal to twice the variance of each order statistic alone. In symbols,

$$S^*_{(np)} - \psi^*_{(np)} = \varepsilon_{np} \quad \text{or} \quad S^*_{(np)} = \psi^*_{(np)} + \varepsilon_{np} \quad , \tag{12}$$

where ε_{np} is asymptotically normal with a mean of zero and a variance that is proportional to $2/n$.

Because (12) holds for every value of np, that is, for every order statistic, we can drop the order-statistic notation. Moreover, as shown earlier, only one of the two variables needs to be transformed, so the general case based on (10) can be written:

$$S = \gamma_S\{\psi\} + \varepsilon \quad . \tag{13}$$

Similarly, the specific case that appears with many distributions, (11), becomes

$$S = \left(\sigma_1 + \sigma_2\psi\right)^{\sigma_3} + \varepsilon' \quad . \tag{14}$$

The variance of ε (or ε') is inversely proportional to sample size, so this error is likely to be small relative to S. This result is quite intuitive; the larger the sample size, the smaller the range over which a given order statistic may vary. As just noted, the sample size here is the number of locations with different values of S, as matched to the number of household types (indicated by ψ). In the Cleveland application in Yinger (2015), for example, high school quality varies over 74 school districts, and some amenities have a separate value for each of 1,665 census block groups. As a result, the error terms in equations (13) and (14) are likely to be small, so that equations (10) and (11) are good approximations for the high-school variable and excellent approximations for these other amenities. Moreover, with a large sample size, the versions of equation (11) that arise from the transformations in Tables 1 to 3 can also be considered to be close approximations. Equations (10) and (11) may not provide good approximations, however, in the case of one-to-one matching with only a few amenity levels and corresponding household types.

As noted earlier, Yinger (2015) draws on the analysis in this paper to derive a form for the hedonic envelope. This derivation is based on (11), not (14). The resulting form is then estimated for cases in which the number of observations is large, such as the school district and block-group examples cited above; in other words, it is only applied to cases in which (11) provides a close approximation to (14). Moreover, the estimating equation for this envelope includes an error term, which captures, among other things, deviations from this derived form. This error term does not correspond exactly to the error term in (14) because it is a summary measure across all amenities, but it allows for random deviations from (11).[14]

When combined with the demand function in Rosen's second step, equation (11) can be used to show the impact of observable household traits on the sorting equilibrium. More specifically, equation (11) can be used to determine the elasticity of S with respect to ψ and the Rosen second step yields the elasticity of ψ with respect to an observable household trait, such as income, Y. The inverse form of the Rosen second step is a regression of the implicit price each household faces on the level of the amenity it receives and its demand traits. The household sets its MWTP equal to the implicit price, so the dependent variable in this regression is actual MWTP. Because the regression holds the amenity level, *S*, constant, however, the estimated elasticity of the dependent variable with respect to *Y* can be interpreted as $(\partial\psi/\partial Y)(Y/\psi)$. Combining these two elasticities reveals the extent to which income (or any other demand trait) determines which households sort into higher-amenity locations:

$$\frac{\partial S}{\partial Y}\frac{Y}{S} = \left(\frac{\partial S}{\partial \psi}\frac{\psi}{S}\right)\left(\frac{\partial \psi}{\partial Y}\frac{Y}{\psi}\right) \qquad (15)$$

Yinger (2015) provides an example using data from the Cleveland area in 2000.[15] The key measure of school quality (= *S*) is the share of students who enter the twelfth grade and subsequently pass all five state tests (mathematics, reading, writing, science, citizenship). The estimates imply that, at the mean values of the variables, $(\partial\psi/\partial Y)(Y/\psi)$ equals 0.623 and that $(\partial S/\partial\psi)(\psi/S)$ equals 0.628. By equation (15), therefore, $(\partial S/\partial Y)(Y/S)$ equals 0.404. In other words, a one percent increase in income leads to a 0.404 percent increase in high school quality solely because of household sorting—that is, because higher-income households win the competition for housing in higher-quality school districts. In the Cleveland data, the mean owner income is $38,136 and the standard deviation, SD, is $15,762. Starting from the mean, a one-SD increase equals an increase of 41.33 percent. The above elasticity indicates that this increase will lead to a (0.404)(41.33) = 16.70 percent increase in school quality, *S*, due to sorting; *S* has a mean of 32.0 and a SD of 20.4, so a 16.7 percent increase is (0.167)(32) = 5.34, or about one-quarter of a standard deviation.

As explained earlier, this analysis sometimes can be extended to consider multiple amenities, with separate income-sorting elasticities, as in (15), for each amenity. If the table of values for two amenities has many empty cells, however, so that certain values of one amenity are only observed at a few values of the other, it may not be possible to analyze sorting for the two amenities separately. If the two amenities are closely related, they can reasonably be combined into a single amenity index with fixed weights. In the case of school quality, for example, many scholars define an index that averages reading and math test scores and perhaps other related variables for a school or school district (Nguyen-Hoang and Yinger 2011). When amenities are not closely related, however, the analysis is more complicated. Epple et al. 2010 define a single linear index of three amenities (a school test score index, the crime rate, and distance to the city center) and estimate a weight for each component. This approach facilitates their general treatment of sorting, which is discussed below, but it also has two disadvantages. First, it omits many other amenities, such as air quality, that have been significant in other hedonic studies and may therefore obtain biased estimates of implicit prices. Second, the linear form of the index is not consistent with the law of diminishing marginal rate of substitution among its components. An alternative approach, found in Yinger (2015) is to estimate the hedonic with a large number of amenity variables and then to conduct sorting analysis based on standard utility postulates and one-to-one matching—but for only a few key amenities. Further research on sorting with multiple amenities would obviously be valuable.[16]

Inexact Matching

Many equilibria are not one-to-one, of course. A given household class may live in more than one location with different levels of the amenity and/or many different household classes may live in a location with a given level of an amenity. As noted earlier, the path-breaking research by Epple and Sieg (1999) and Epple et al. (2001) accounts for these possibilities by solving and estimating a general equilibrium model of bidding and sorting using community-level data. Moreover, the empirical analysis by Epple et al. (2010), which is based on house sales data from Pittsburgh, incorporates this type of general model. This study provides a table indicating, among other things, that “High-income households with children have stronger tastes for high price (and high amenity) communities than households without children” (p. 216). However, this approach does not provide a simple way to summarize the relationship between household preferences and amenity levels. Moreover, as noted above, his approach builds on the strong assumption that there is a single amenity index.

A well-known example of across-location homogeneity arises when all households are alike. In this case, the hedonic equilibrium is a bid function, and the envelope concept is not needed. Household heterogeneity appears to be far more extensive than amenity heterogeneity, however. A more realistic possibility is plotted in Figure 2. The plotted envelope assumes one-to-one matching. The steepest bid-function in the figure is tangent to the envelope at a passing rate of 0.26. If no other households have steeper bid functions and another location has a passing rate of 0.3, then, assuming that this picture represents an equilibrium, the household class with this bid function will also win the competition for housing in this other jurisdiction. Above a passing rate of 0.26, therefore, observed prices will follow this household class’s bid function, not the derived bid-function envelope.

Because the envelope and the bid function have similar slopes above this point, the envelope is likely to provide a reasonable approximation to the bid-function in this case, at least if the locations involved do not exhibit a large range in the amenity. In other words, the assumption of one-to-one matching may be a reasonable approximation to cases of across-location heterogeneity so long as a household class, that is, households with the same relative MWTP, does not live in a set of communities with widely differing levels of the amenity.[17]

An important example of the second possibility, within-location heterogeneity, is that of a big city in an analysis of a city-level amenity, such as school-district quality, with heterogeneous households. In their treatment of this issue, Epple and Sieg (1999) and Epple et al. (2001) do not formally define ψ; instead their bid function slopes depend on income and a random taste parameter with an assumed joint distribution. They show that the housing price difference between a big city and the jurisdiction ranked one step higher in service level is determined by the city household class with the steepest bid function, called the marginal household. A higher price in the city would induce the marginal household to leave, and a lower price would attract too many households.

In contrast, a functional form based on one-to-one matching assumes that the housing price difference between these two service levels reflects both bidding and sorting. Because some city household classes have flatter bid functions than the marginal household, the predicted difference in the envelope between these two service levels with sorting understates the difference along the marginal household's bid function. However, the bid function of the marginal household is tangent to the envelope in the jurisdiction with a somewhat higher amenity level than the city, so the slope of this household's bid function and the slope of a functional form that assumes one-to-one matching may not diverge very much unless these two jurisdictions have significantly different amenity levels. If the two amenity levels are similar, this type of clustering limits the variation in S that is available to help estimate the envelope but may not be a significant source of bias in the estimated envelope parameters.

These points are illustrated in Figure 2. Consider a large city with an amenity level (e.g. school passing rate) of 0.15, the lowest in the area, and another district with the next-lowest rate of 0.2. Then the observed price when the amenity equals 0.15 is on the dotted line, not on the envelope. This dotted line is the marginal household's bid function; it is tangent to this envelope just below the point where the amenity equals 0.2.

In short, deviations from one-to-one matching result in segments along which observed housing prices follow a bid function, not the envelope. Because bid functions are tangent to the envelope at the edges of these segments, the resulting deviations from the envelope may be small, at least when these segments involve small ranges in the amenity level. Under many circumstances, therefore, a form based on one-to-one matching may provide a reasonable approximation to the actual market price function—even with big cities. More research on the nature of this approximation is needed.

Conclusions

Households must compete for housing in locations with high-quality amenities. The outcome of this competition can be described by the relationship between the amenity level the household receives and its relative marginal willingness to pay for the amenity. This paper shows that this equilibrium relationship reflects the distributions of the amenity and of households' relative MWTP and derives expressions for this equilibrium under various assumptions about these distributions—in the case of one-to-one matching. One key result is that if a monotonic transformation of the amenity has the same distribution as a monotonic transformation of relative bid-function slopes, then the equilibrium relationship between the amenity and the slope depends only on these transformations, not on the post-transformation distribution. Linear transformations, for example, lead to a linear equilibrium relationship between these two variables.

The analysis in this paper assumes that housing-market sorting can be characterized by one-to-one matching, defined as an equilibrium in which every household class, identified by its relative MWTP, receives a unique value of an amenity. One-to-one matching models use continuous mathematics to approximate the allocation of households to locations, and empirical applications make use of the asymptotic properties of the relevant distributions. These models are obviously not appropriate for analyzing household allocation for amenities with little variation in an urban area or with extensive within-jurisdiction variation in household types, but they appear to provide reasonable approximations under many other circumstances.

The analysis in this paper leads to several issues for future research on sorting. First, it would be useful to compare models based on one-to-one matching, which require strong assumptions about matching but do not require strong assumptions about amenity demand or household heterogeneity, and models based on inexact matching, which provide a more general treatment of matching but currently require a single amenity index and a specific parameterization of household heterogeneity.[18] Second, hedonic forms derived using equation (11) make it possible to determine whether and to what extent the sorting equilibrium, which is summarized by the σ parameters, changes over time. The time pattern of these parameters could shed light on the extent and causes of household re-sorting. Third, although many assumptions about the distributions of S and ψ lead to equation (11), other assumptions lead to different forms for the sorting equilibrium. An investigation of different forms for the sorting equilibrium, along with associated estimating equations, would also be valuable.

Table 1. Transformations for S and ψ that Lead to an Equilibrium Consistent with Equation (11), with Indicated Constraints

	Transformation for *S*			
Transformation for ψ	None	Linear	n-th Degree Power Function	Same Form as Inverse of Equation (11), exponent=$1/s \neq 1$
None	$\sigma_1 = 0$; $\sigma_2 = 1$; $\sigma_3 = 1$	$\sigma_1 > 0$; $\sigma_2 > 0$; $\sigma_3 = 1$	$\sigma_1 = 0$; $\sigma_2 > 0$; $\sigma_3 = 1/n$	$\sigma_1 > 0$; $\sigma_2 > 0$; $\sigma_3 = s$
Linear	$\sigma_1 > 0$; $\sigma_2 > 0$; $\sigma_3 = 1$	$\sigma_1 > 0$; $\sigma_2 > 0$; $\sigma_3 = 1$	$\sigma_1 > 0$; $\sigma_2 > 0$; $\sigma_3 = 1/n$	$\sigma_1 > 0$; $\sigma_2 > 0$; $\sigma_3 = s$
m-th Degree Power Function	$\sigma_1 = 0$; $\sigma_2 > 0$; $\sigma_3 = m$	N.A.	$\sigma_1 = 0$; $\sigma_2 > 0$; $\sigma_3 = m/n$	N.A.
Same Form as Equation (11), exponent = $s \neq 1$	$\sigma_1 > 0$; $\sigma_2 > 0$; $\sigma_3 = s$	N.A.	$\sigma_1 > 0$; $\sigma_2 > 0$; $\sigma_3 = s/n$	N.A.
Notes: All the coefficients in each of the transformations are assumed to be positive; N.A. indicates that the two transformations do not lead to a form that is consistent with Equation (11), usually because of an extra additive constant; an n-th degree power function is defined as $y = cx^n$, with $n \neq 1$.				

Table 2. Transformations from One Distribution to Another that Has the Same Form but Different Parameters that Result in an Equilibrium Consistent with Equation (11)

Distribution for *S* and ψ	**Restrictions on Equation (11)**
Distributions that Can Be Scaled (= Proportional Transformation)	
Gamma	$\sigma_1 = 0$; $\sigma_2 > 0$; $\sigma_3 = 1$
Exponential	$\sigma_1 = 0$; $\sigma_2 > 0$; $\sigma_3 = 1$
Erlang	$\sigma_1 = 0$; $\sigma_2 > 0$; $\sigma_3 = 1$
Half Normal	$\sigma_1 = 0$; $\sigma_2 > 0$; $\sigma_3 = 1$
Weibull	$\sigma_1 = 0$; $\sigma_2 > 0$; $\sigma_3 = 1$
Distributions that Can Be Scaled and Relocated (= Linear Transformation)	
Fréchet	$\sigma_1 > 0$; $\sigma_2 > 0$; $\sigma_3 = 1$
Lévy	$\sigma_1 > 0$; $\sigma_2 > 0$; $\sigma_3 = 1$
Truncated Normal (positive domain)[a]	$\sigma_1 > 0$; $\sigma_2 > 0$; $\sigma_3 = 1$
Raleigh	$\sigma_1 > 0$; $\sigma_2 > 0$; $\sigma_3 = 1$
Distributions that Can be Raised to a Power (= Power Transformation)	
Log-Normal	$\sigma_1 = 0$; $\sigma_2 = 1$; $\sigma_3 > 0$

These are examples drawn from the set of distributions with positive domain. Information on distributions comes from Leemis and McQueston (2008) and Wolfram Research (2015). Distributions in the second and third panels also fit into the first panel for certain parameter values in the transformation.

[a] This case only applies if the post-transformation truncation points are the same in the two distributions.

Table 3. Transformations from One Distribution to Another that Yield an Equilibrium Consistent with Equation (11)

Distribution for *S*	**Distribution for ψ**	**Restrictions on Equation (11)**
Maxwell-Boltzmann{*a*}	Gamma{3/2, 2a^2}	$\sigma_1 = 0$; $\sigma_2 = 1$; $\sigma_3 = 1/2$
Gamma{3/2, 2a^2}	Maxwell-Boltzmann{*a*}	$\sigma_1 = 0$; $\sigma_2 = 1$; $\sigma_3 = 2$
Standard Uniform (= (0, 1) interval)	Beta{1/*a*,1}	$\sigma_1 = 0$; $\sigma_2 = 1$; $\sigma_3 = 1/a$, $a>0$
Beta{1/*a*,1}	Standard Uniform (= (0,1) interval)	$\sigma_1 = 0$; $\sigma_2 = 1$; $\sigma_3 = a$, $a>0$
Exponential{λ}	Raleigh{*s*}	$\sigma_1 = 0$; $\sigma_2 = 1/(s\sqrt{2\lambda})$; $\sigma_3 = 2$
Raleigh{*s*}	Exponential{λ}	$\sigma_1 = 0$; $\sigma_2 = 2s^2\lambda$; $\sigma_3 = 1/2$
Exponential{*1*}	Weibull{λ, *a*}	$\sigma_1 = 0$; $\sigma_2 = 1/\lambda$; $\sigma_3 = a$
Weibull{λ, *a*}	Exponential{*1*}	$\sigma_1 = 0$; $\sigma_2 = \lambda^a$; $\sigma_3 = 1/a$
These are examples drawn from the set of distributions with positive domain. Information on distributions comes from Leemis and McQueston (2008) and Wolfram Research (2015).		

Figure 1. Bidding and Sorting

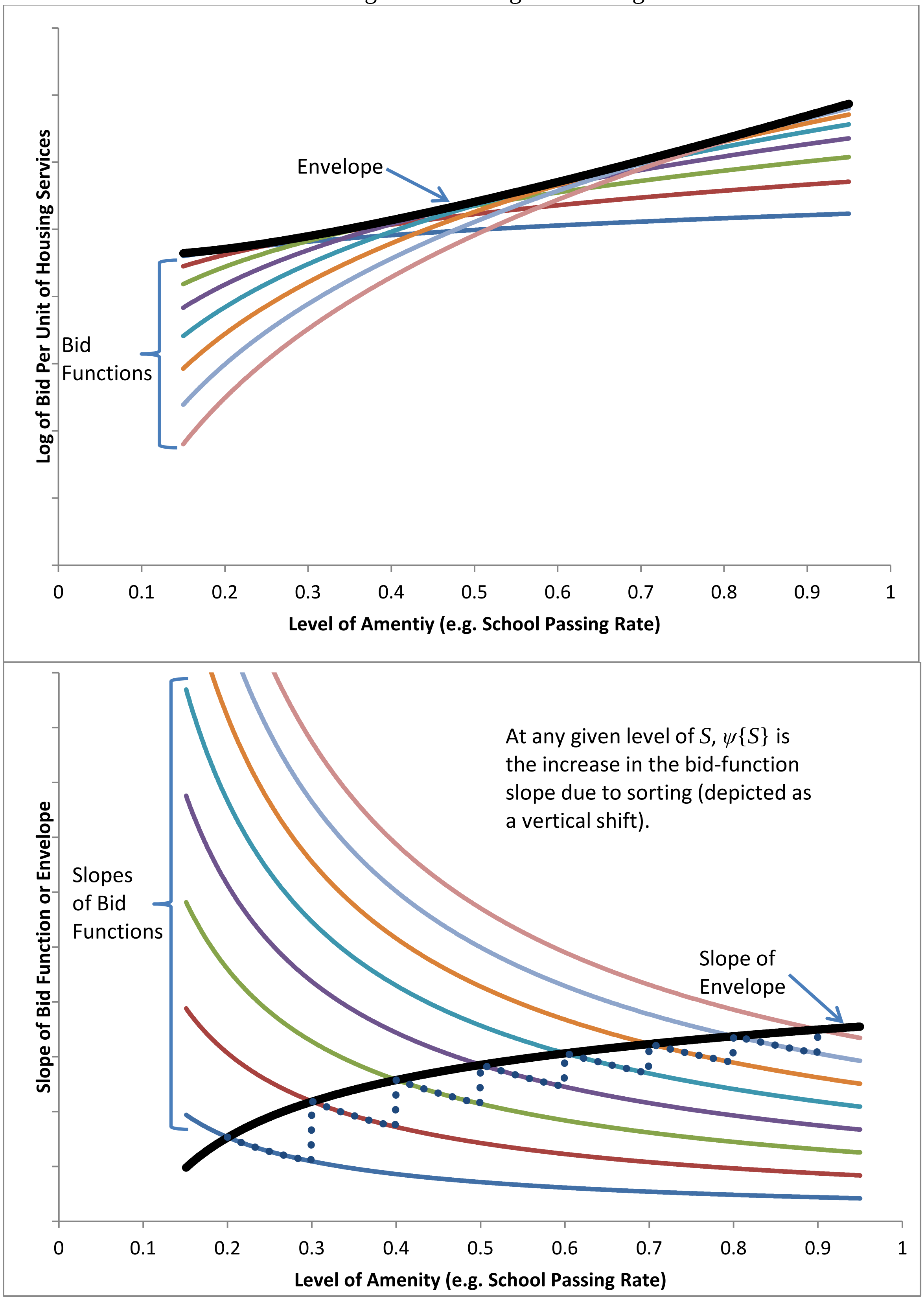
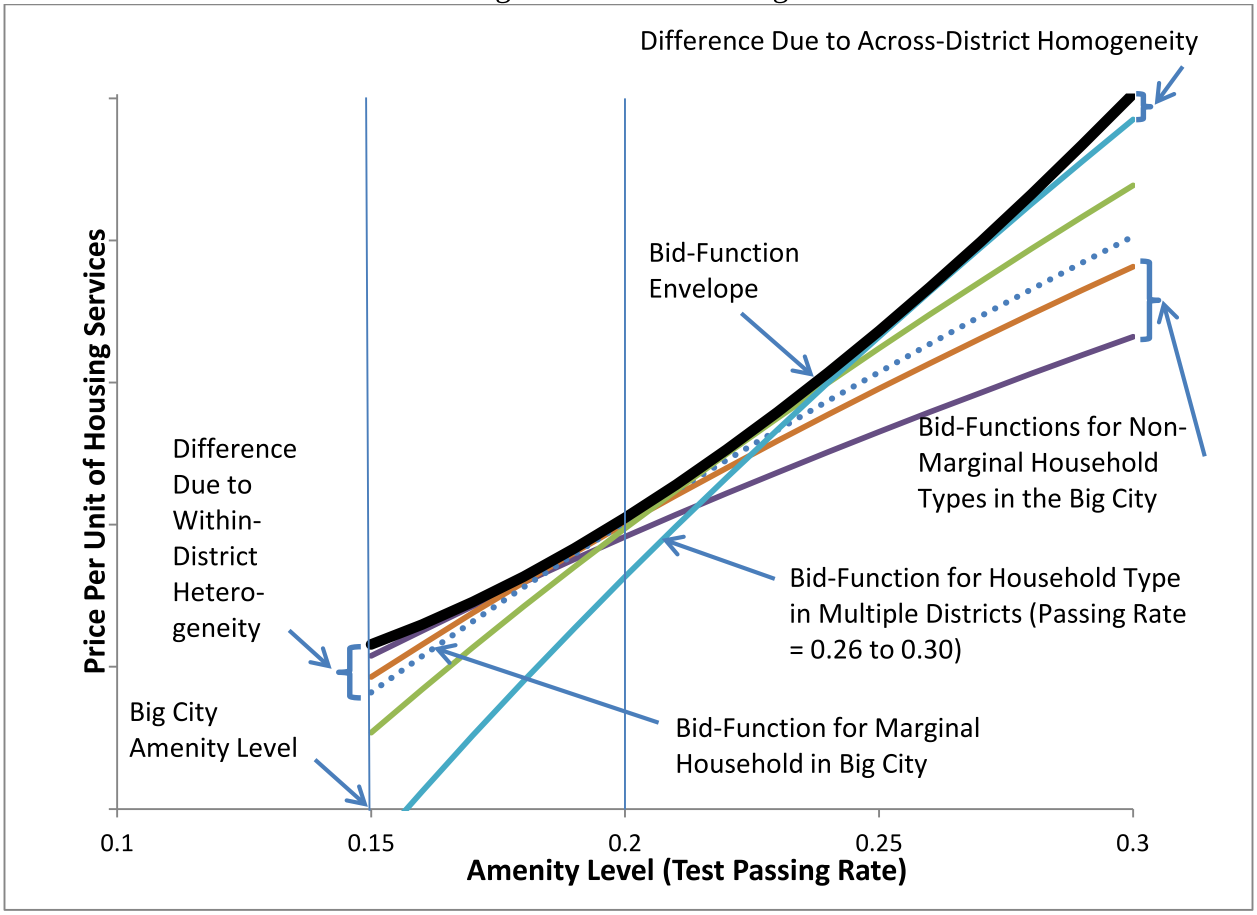


Figure 2. Inexact Sorting



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Endnotes

[1] For literature reviews, see Sheppard (1999) and Taylor (2008) on hedonics and Ross and Yinger (1999) on local public finance. Tiebout (1956) is a key paper in local public finance; Rosen (1974) recognized that his model was "similar in spirit to Tiebout's (1956) analysis of the implicit market for neighborhoods, local public goods being the 'characteristics' in this case. He obtained the result that neighborhoods tend to be segmented by distinct income and taste groups" (p. 40). Rosen (p. 38) also acknowledges his debt to Alonso (1964) who developed a version of the famous graph, complete with bidding and sorting, for the amenity "access to jobs."

[2] Taylor (2008, p. 20) argues that "the hedonic price function is linear" when "the product can be costlessly repackaged." Alternative approaches to structural housing traits are reviewed in Sheppard (1999) and Taylor (2008).

[3] Households may bid on the basis of anticipated changes in amenity levels due, say, to extensive demographic change or the construction of a new park. Amenity supply is still exogenous in this case, although amenities may be difficult to measure. The assumption of exogenous supply is not appropriate, however, for developer-supplied amenities, such as those in a gated community.

[4] This figure is also found in Yinger (2015). A figure comparable to Panel B appears in Rosen (1974, Fig. 4)—but without a decomposition of the bidding and sorting components of the slope.

[5] School capitalization studies are reviewed in Nguyen-Hoang and Yinger (2011).

[6] Rosen's second step faces a well-known endogeneity problem: the simultaneous determination of implicit prices and amenity levels. See Taylor (2008). Endogeneity is not inherent in Rosen's first step, however. The observable and unobservable determinants of a given household's bids do not appear in the hedonic envelope; they are integrated out in the sorting process. Amenities may, of course, be endogenous in other contexts. Long-run sorting outcomes for neighborhood

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ethnicity, for example, must be consistent with the ethnic compositions on which bids are based (Yinger 1976). Moreover, with jurisdiction-level data, equilibrium requires the public service outcomes on which bids are based to be consistent with the outcomes selected by the voters in the jurisdiction; Epple, Romer, and Sieg (2001) provide an insightful treatment of this case.

[7] Several other recent approaches to hedonics are discussed in Yinger (2015); in addition, Bayer, Ferreira, and McMillan (2007) derive a discrete-choice model to analyze sorting equilibria.

[8] The associated pre-property-tax bid function is $\hat{P}\{S\}^{(1+\nu)} = C + \psi S^{([1+\mu]/\mu)}$, where ν (μ) is the price elasticity of demand for H (S), C is a constant, and parentheses in an exponent indicate the Box-Cox form. This form can be derived from an indirect utility function (Yinger 2015), which is similar to the one in Epple et al (2010), except that it includes latent demand functions for several amenities, instead of the level of one amenity index.

[9] Imagine this two-quadrant diagram: S is measured upward from the origin on the y-axis, and α is measured downward; ψ is measured on the x-axis. The bottom quadrant plots $\psi_\alpha\{\alpha\}$, which obviously depends on the distribution of ψ, and the top quadrant plots $S = g_S^{-1}\{g_\psi\{\psi\}\}$. Starting from any value of α, one can first find the associated ψ in the bottom quadrant and then find S in the top quadrant. This step corresponds to equation (10).

[10] An equivalent statement is that if both S and ψ have normal distributions, then separate linear transformations can turn each distribution into a standard normal. If S and ψ have truncated normal distributions, then they can still be linked by a linear transformation—so long as the post-transformation truncation boundaries are the same.

[11] The assumption that the demand for S_i does not depends on S_j may seem strong, but it greatly simplifies the derivation of envelopes and, except in special cases, it is needed to preserve the single-crossing condition. Yinger (2015) shows, however, that the demand for S_i can depend on

variables predicting S_j. If the error terms in two demand equations are correlated, then seemingly unrelated regression techniques are needed to obtain efficient estimates of demand parameters.

[12] Yinger (2015, pp. 11-12) refers to the exponential-Weibull case in Table 3, but this reference fails to indicate that this case involves the standard exponential distribution and it incorrectly says $\sigma_2 = 1$ instead of $\sigma_2 = 1/\lambda$.

[13] If $(n)(p)$ is not an integer, Mood, Graybill, and Boes replace p with p_n, "such that np_n is an integer and $n|p_n - p|$ is bounded" (p. 257). In this case the distribution they present is for "approximately the (np)th order statistic."

[14] Yinger's (2015) envelope derivation involves an integral, which cannot be solved when an error term is present in (11). Moreover, because the estimated hedonic covers all neighborhood amenities, a separate error term for each amenity could not be estimated.

[15] By using equation (11), Yinger (2015) derives a second step in which ψ, not the implicit price, is regressed on demand traits. This approach also yields $\partial\psi/\partial Y$, and it avoids the endogeneity problem in the Rosen second step.

[16] One promising extension would build on the combination of factor analysis and hedonics, which was introduced by Kain and Quigley (1970). This approach would find patterns in the neighborhood amenities using factor analysis and then apply the methods in this paper and in Yinger (2015) to the presumably reduced number of neighborhood factors.

[17] The closeness of the approximation obviously depends on the shape of the envelope. With a steeply rising envelope, for example, the approximation may not be so good.

[18] As shown by Yinger (2015), a model based on one-to-one matching also leads to a simpler estimating procedure for the hedonic regression than does a model with inexact sorting.