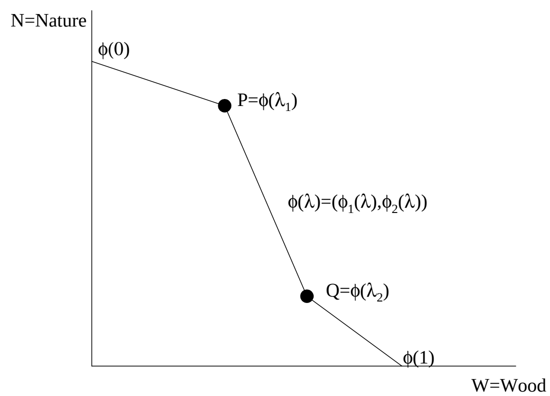
ECN 612 - HANDOUT - FALL 2011
Chapter 3: National Forest

SECTION 3.8: A NATIONAL FOREST

The model in this chapter is somewhat different from the one in Chapter 1. Let $\phi : [0,1] \to \mathbf{R}^2_+$ be a parametric equation describing the levels of wood and nature produced for varying intensity of logging in a forest, i.e.,

$$\phi(\lambda) = (\phi_1(\lambda), \phi_2(\lambda)),$$

where $\phi_1(\lambda)$ is the amount of wood (W) produced and $\phi_2(\lambda)$ is the level of nature (N) preserved when the intensity of logging is λ. We assume that the technology is represented in the following diagram. Note that as λ increases, W increases and N decreases. There are kinks at $\lambda = \lambda_1$ and $\lambda = \lambda_2$.



Assume that Agent 1 represents the loggers and Agent 2 represents the preservationists. Each has an associated political action function $P_i : [0,1] \times \Theta^i \to \mathbf{R}_+$, $i = 1, 2$ (shown in the graph below). These represent the levels of political pressure applied by these agents, which depend on their characteristics and the intensity level of logging. More precisely,

$$P_1(\lambda, a) = P_1(\lambda, a_0, a_1, a, a_3),\ P_2(\lambda, b) = P_2(\lambda, b_0, b_1, b_2, b_3),$$

where $a = (a_0, a_1, a, a_3)$ is the vector of characteristics of Agent 1 and $b = (b_0, b_1, b_2, b_3)$ is the vector of characteristics of Agent 2. We assume that P_1 is strictly decreasing with respect to λ (as the logging intensity increases, the political pressure applied by the loggers decreases), and P_2 is strictly increasing with respect to λ (as the logging intensity increases, the political pressure applied by the preservationists increases). Also,

$$P_1(0, a) = a_0,\ P_1(1, a) = a_4,\ P_2(0, b) = b_0,\ P_2(1, b) = b_4$$

for all $a \in \Theta^1$ and all $b \in \Theta^2$. The graphs of both P_1 and P_2 (as functions of λ) are made up of line segments. More specifically, the graph of $P_1(\lambda, a_0, a_1, a, a_3)$ (as a function of λ) is made up of line segments between the following pairs of points:

$$\{((0, a_0), (\lambda_1, a_1)), ((\lambda_1, a_1), (\lambda_2, a_2)), ((\lambda_2, a_2), (1, a_3))\};$$

similarly, $P_2(\lambda, b_0, b_1, b_2, b_3)$ (as a function of λ) is made up of line segments between the following pairs of points:

$$\{((0, b_0), (\lambda_1, b_1)), ((\lambda_1, b_1), (\lambda_2, b_2)), ((\lambda_2, b_2), (1, b_3))\}.$$

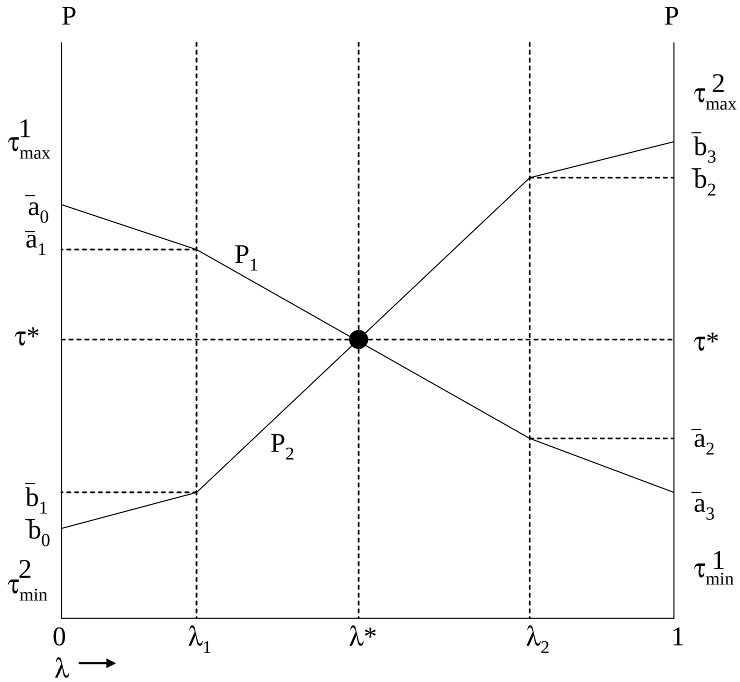
It is assumed throughout that

$$\Theta^1 = \left\{(a_0, a_1, a, a_3) : \tau^1_{\min} \le a_3 < a_2 < a_1 < a_0 \le \tau^1_{\max}\right\}$$

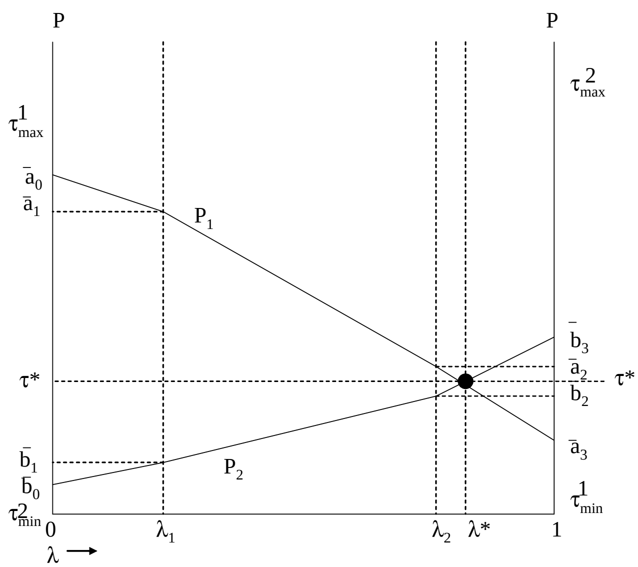
and

$$\Theta^2 = \left\{(b_0, b_1, b_2, b_3) : \tau^2_{\min} \le b_0 < b_1 < b_2 < b_3 \le \tau^2_{\max}\right\}.$$

We also assume that $\left[\tau^1_{\min}, \tau^1_{\max}\right] \cap \left[\tau^2_{\min}, \tau^2_{\max}\right] \ne \varnothing$ (this guarantees that there exist P_1 and P_2 which intersect). The political pressure functions are represented in the following diagram, where $a = \bar{a}$, $b = \bar{b}$, and $\lambda_1 < \lambda^* < \lambda_2$.



Another case, where $\lambda_2 < \lambda^*$, is represented below, although we will not analyze this and other cases.



We assume that the Forester chooses that level of intensity at which the political pressure applied by each agent is the same. That is, the goal function is

$$F\left(\bar{a}, \bar{b}\right) = \lambda^*. \tag{1}$$

However, the Forester does not know the characteristics of the agents, and, therefore, cannot determine directly λ^*, the intensity level of logging at which political pressures are equal.

CONSTRUCTING A MECHANISM REALIZING F USING THE FORMAL APPROACH

CONSTRUCTING THE rRM CORRESPONDENCE V(θ)

Select a base point $\bar{\theta} = \left(\bar{\theta}^1, \bar{\theta}^2\right) = \left(\bar{a}, \bar{b}\right) = \left(\bar{a}_0, \bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{b}_0, \bar{b}_1, \bar{b}_2, \bar{b}_3\right) \in \Theta^1 \times \Theta^2$. Assume that $F\left(\bar{\theta}\right) = \lambda^*$. Denote by (λ^*, τ^*) the point at which $P_1\left(\lambda, \bar{a}\right)$ and $P_2\left(\lambda, \bar{b}\right)$ intersect. Set $B_1 = \left\{\bar{\theta}^2\right\}$.

1. As the first step in the construction of a largest rectangle contained in $F^{-1}\left(F\left(\bar{\theta}\right)\right) = F^{-1}\left(\lambda^*\right)$, find the largest θ^1 side for the rectangle, denoted by $A^*\left(B_1, \bar{\theta}\right)$, so that $\bar{\theta}^1 \in A_1$ and so that the rectangle $A_1 = A^*\left(B_1, \bar{\theta}\right) \times \bar{\theta}^2 \subseteq F^{-1}\left(F\left(\bar{\theta}\right)\right) = F^{-1}\left(\lambda^*\right)$ (this means the rectangle is contained in the level set associated with the outcome to which $\bar{\theta}$ is assigned).
2. Now construct the largest θ^2 side for the rectangle, denoted by $B_2 = B^*\left(A_1, \bar{\theta}\right)$, so that the set is the largest possible subject to the constraints that $\bar{\theta}^2 \in B_2$ and the rectangle $A_1 \times B^*\left(A_1, \bar{\theta}\right) \subseteq F^{-1}\left(F\left(\bar{\theta}\right)\right)$.

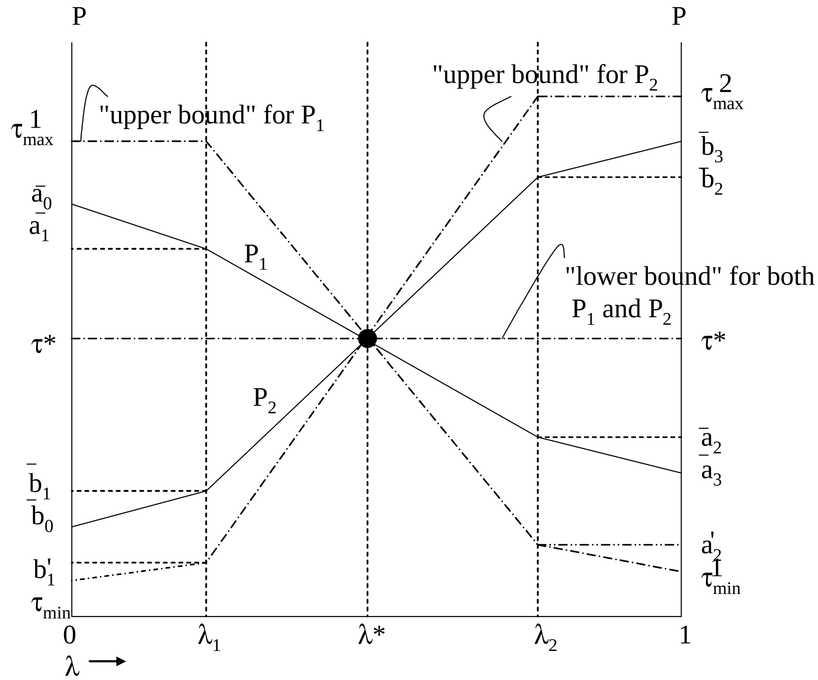
One might think we need a third step, taking B_2 and finding the the largest θ^1 side for the rectangle associated with it, denoted by A_2. However, by Thm 3.4.1, p. 191, the side found in Step 1 is the largest θ^1 side for the rectangle given B_2. Define the rectangle as

$$V\left(\bar{\theta}\right) = \tilde{A}\left(\bar{\theta}\right) \times \tilde{B}\left(\bar{\theta}\right),$$

where

$$\tilde{A}\left(\bar{\theta}\right) = A_1 = A^*\left(\tilde{B}\left(\bar{\theta}\right), \bar{\theta}\right),\ \tilde{B}\left(\bar{\theta}\right) = B_2 = B^*\left(\tilde{A}\left(\bar{\theta}\right), \bar{\theta}\right).$$

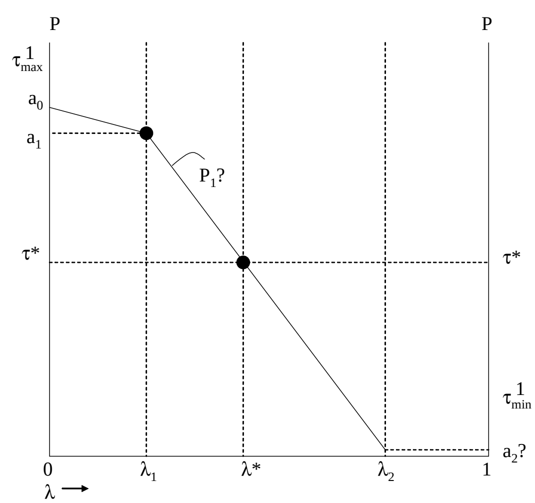
We illustrate how to find $V\left(\bar{\theta}\right) = \tilde{A}\left(\bar{\theta}\right) \times \tilde{B}\left(\bar{\theta}\right)$. Notice that there is a lot of leeway with respect to what P_1 and P_2 functions will ultimately yield an outcome of λ^*. But given the base point $\left(\bar{a}, \bar{b}\right)$, any P_1 and P_2 functions allowed must go through (λ^*, τ^*). To see this, note given that $\bar{a} \in \tilde{A}\left(\bar{\theta}\right)$, any P_2 function allowed must go through (λ^*, τ^*) (otherwise, $P_1\left(\lambda^*, \bar{a}\right)$ and $P_2\left(\lambda^*, b\right)$ will not intersect at λ^* since $P_1\left(\lambda^*, \bar{a}\right) = \tau^*$). Also, given that $\bar{b} \in \tilde{B}\left(\bar{\theta}\right)$, any P_1 function allowed must go through (λ^*, τ^*) (otherwise, $P_1\left(\lambda^*, a\right)$ and $P_2\left(\lambda^*, \bar{b}\right)$ will not intersect at λ^* since $P_2\left(\lambda^*, \bar{b}\right) = \tau^*$). So, assume that τ is fixed at τ^*. Then we find all (a_0, a_1, a_2, a_3) and (b_0, b_1, b_2, b_3) and their associated P_1 and P_2 functions so that the functions intersect at (λ^*, τ^*). In the following diagram, we illustrate how we can alter the P_1 and P_2 functions and still retain the intersection at (λ^*, τ^*).



Following the development in Chapter 1, notice that if we set a_1 and constrain P_1 to satisfy its required properties (and go through the point (λ^*, τ^*)), then a_2 is determined. More precisely, it can be shown that

$$a_2 = \frac{\tau^* - \mu a_1}{1 - \mu}, \text{ where } \mu = \frac{\lambda - \lambda_1}{\lambda_1 - \lambda_2}$$

(see p. 59 of your text), where we need to place appropriate restrictions. We cannot choose $a_1 < \tau^1_{\max}$ to be too large because that would mean that a_2 would be too small (the constraint that $\tau^1_{\min} < a_2$ would be violated). This is illustrated in the diagram below.



Summarizing the previous paragraph, we write

$$a_2 = \bar{\xi}_1 \left(a_1, \bar{\theta}\right),$$

where we now define appropriate restrictions on a_1. Define

$$a_2 = \xi_1 \left(a_1, \bar{\theta}\right),$$

where the latter function is defined on the set

$$D_1\left(\bar{\theta}\right) = \left\{a_1 : a_1 < \tau^1_{\max} \text{ and } a_2 = \bar{\xi}_1\left(a_1, \bar{\theta}\right) > \tau^1_{\min}\right\}.$$

We find that the set

$$\tilde{A}\left(\bar{\theta}\right) = \left\{(a_0, a_1, a_2, a_3) \in \Theta^1 : (a_1, a_2) = \left(a_1, \xi_1\left(a_1, \bar{\theta}\right)\right), \text{ where } a_1 \in D_1\left(\bar{\theta}\right)\right\}$$

is the largest set contained in Θ^1 such that the P_1 function goes through the point (λ^*, τ^*). (Notice that I have altered the notation somewhat from Chapter 1. Specifically, $\xi_1\left(a_1, \bar{\theta}\right)$ and $D_1\left(\bar{\theta}\right)$ replace $\xi_1\left(a_1, (\lambda^*, \tau^*)\right)$ and $D_1(\lambda^*, \tau^*)$, respectively, since λ^* and τ^* are determined by $\bar{\theta}$. I have made similar changes for ξ_2 and D_2 below.)

We can similarly change the P_2 function and still retain the intersection at (λ^*, τ^*). I will leave you lay out a similar way of defining $\bar{\xi}_2$, ξ_2, and D_2. The set

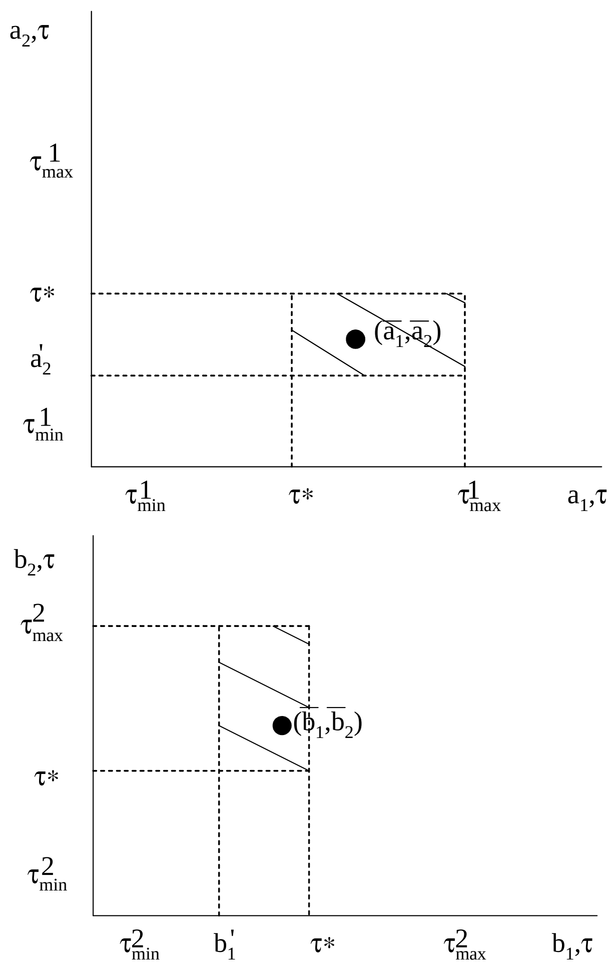
$$\tilde{B}\left(\bar{\theta}\right) = \left\{(b_0, b_1, b_2, b_3) \in \Theta^2 : (b_1, b_2) = \left(b_1, \xi_2\left(b_1, \bar{\theta}\right),\right), \text{ where } b_1 \in D_2\left(\bar{\theta}\right)\right\}$$

is defined in a way analogous to the set $\tilde{A}\left(\bar{\theta}\right)$.

Notice that the set $V\left(\bar{\theta}\right) = \tilde{A}\left(\bar{\theta}\right) \times \tilde{B}\left(\bar{\theta}\right)$ has the following properties:

1. $\bar{\theta} \in V\left(\bar{\theta}\right)$,
2. it is a rectangular subset of the parameter space $\Theta = \Theta^1 \times \Theta^2$,
3. it is contained in the level set $F^{-1}(\lambda^*)$,
4. there is no subset of that has properties 1 and 2 and contains $V\left(\bar{\theta}\right) = \tilde{A}\left(\bar{\theta}\right) \times \tilde{B}\left(\bar{\theta}\right)$ as a proper subset.

The second and third components of the elements of $\tilde{A}\left(\bar{\theta}\right)$ and $\tilde{B}\left(\bar{\theta}\right)$ are represented in the following diagrams, where $a_2^{'} = \xi_1\left(\tau^1_{\max}, \bar{\theta}\right)$ and $b_1^{'} = \xi_2\left(\tau^2_{\max}, \bar{\theta}\right)$.



CONSTRUCTING THE SYSTEM OF DISTINCT REPRESENTATIVES

Notice that the covering

$$\mathcal{C} = \mathcal{C}_V = \{V(\theta) : \theta \in \Theta\},$$

of Θ is a partition of Θ (I will leave this for you to show). Therefore, we can define the system of distinct representatives as $\Lambda : \mathcal{C} \to \Theta$ by $\Lambda(V(\theta)) = \theta = (a, b)$.

APPLYING AN ENCODING FUNCTION

We can define the encoding function $\nu : \Lambda(\mathcal{C}) \to M$ by $\nu(\theta) = (\lambda, \tau)$, where (λ, τ) are such that $P_1(\lambda, a) = \tau = P_2(\lambda, b)$. (It is a one-to-one function - why?) The encoding function reduces the transversal from a space of dimension 8 to the message space of dimension 2. The message space has elements of the form

$$(m_1, m_2) = (\lambda, \tau).$$

We can define $M = [0, 1] \times \mathbf{R}_+$, and we let

$$\mu^1(a) = \{(m_1, m_2) \in M : P_1(m_1, a) = m_2\},$$

$$\mu^2(b) = \{(m_1, m_2) \in M : P_2(m_1, b) = m_2\}.$$

Define

$$\mu(a, b) = \mu^1(a) \cap \mu^2(b)$$

and $M' = \mu^1(\Theta^1) \cap \mu^2(\Theta^2)$. Notice that

$$\mu(a, b) = \{(m_1, m_2) \in M : P_1(m_1, a) = P_2(m_1, b) = m_2\}.$$

For $(m_1, m_2) \in \mu(a, b)$, define $h(m_1, m_2) = m_1 = \lambda^*$. The mechanism $\pi = (\mu, M', h)$ realizes the goal function F defined at (1).

FOR YOU

I will leave it for you to see the discussion on p. 271 about increasing the number of parameters characterizing the P_i, $i = 1, 2$, functions without increasing the size of the message space.

REFERENCES

Hurwicz and Reiter.