ECN 612 - HANDOUT - FALL 2009
Chapter 3: Part 1

Notation, Definitions

- $F : \Theta \to Z$ is the goal function, where
  - Θ is the environment space
  - Z is the outcome space
- $\pi = (M, \mu, h)$ is a *mechanism,* where
  - M is the message space; contains messages available for communication
  - $\mu : \Theta \rightrightarrows M$ is the (equilibrium) message correspondence
  - $h : M \to Z$ is the outcome function

The mechanism when operated in the environment θ yields outcomes $h(\mu(\theta))$ in Z.

Definition 1 *The mechanism $\pi = (M, \mu, h)$ realizes F if for all $\theta \in \Theta$*

$$h \circ \mu(\theta) = F(\theta).$$

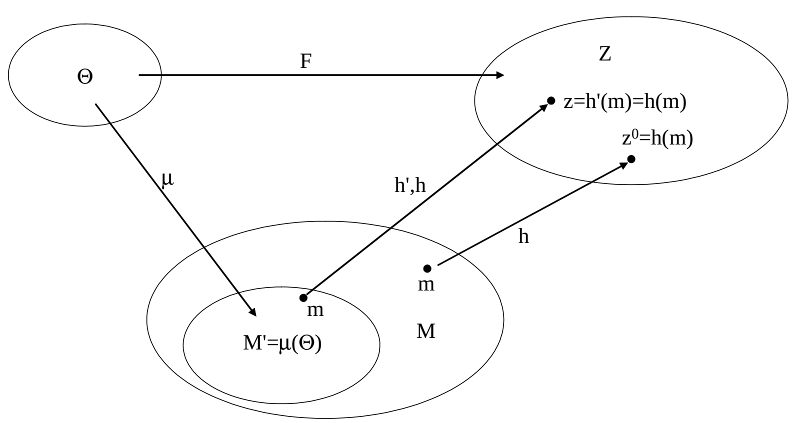
This idea is represented in the following diagram.

$$\begin{array}{ccccc} \Theta & & \overset{F}{\to} & & Z \\ & \underset{\mu}{\searrow} & & \underset{h}{\nearrow} & \\ & & M & & \end{array}$$

Note We also distinguish between a mechanism $\pi' = (M', \mu, h')$ and $\pi = (M, \mu, h)$, where $M' \subseteq M$, where $M' = \mu(\Theta)$ is the image of Θ under μ (M' is the set of messages in M that are actually used as part of the mechanism). Note that a mechanism $\pi' = (M', \mu, h')$, where $M' = \mu(\Theta)$, and $h' : M' \to Z$, can be extended to $\pi = (M, \mu, h)$ by defining $h : M \to Z$ by

$$h(m) = \begin{cases} h'(m) & \text{if } m \in M' \\ z^0 \in Z & \text{if } m \in M \backslash M', \end{cases}$$

where $z^0 \in Z$ is arbitrary.



Definition 2 *Assume that the set of agents is $I = \{1, \ldots, N\}$ and that*

$$\Theta = \Theta^1 \times \cdots \times \Theta^N. \tag{1}$$

If F is the goal function, F is factored.

Definition 3 *A mechanism $\pi = (M, \mu, h)$ is (informationally) decentralized (or privacy preserving) if and only if there exist correspondences $\mu^i : \Theta^i \rightrightarrows M$ such that for all $\theta \in \Theta$, $\mu(\theta) = \cap_{i=1}^N \mu^i(\theta^i)$.*

Motivation for Rectangles Method

Assume that the mechanism $\pi = (M, \mu, h)$ realizes $F : \Theta = \Theta^1 \times \cdots \times \Theta^N \to Z$. Note that the (equilibrium) message correspondence $\mu : \Theta \Rightarrow M$ induces a covering C_μ of the parameter space Θ where each set in C_μ is rectangular and $F - cc$. To see this, note first that $\mu^{-1} : M' \Rightarrow \Theta$ is defined in the following way:

$$\mu^{-1}(m) = \{\theta \in \Theta : \mu(\theta) = m\}.$$

Define

$$K(m) = \mu^{-1}(m),$$

and

$$C_\mu = \{K(m) : m \in M'\}.$$

I will leave you to show that C_μ is covering of the parameter space Θ (you need to show that $\cup_{K(m)\in C_\mu} K(m) = \Theta$).

Now we show that for every $K(m) \in C_\mu$, $K(m) \subseteq F^{-1}(F(\theta))$, i.e., $K(m)$ is $F - cc$ ($F-$contour contained). Since $\pi = (M, \mu, h)$ realizes $F : \Theta \to Z$, for every $m \in M'$and every $\theta \in \mu^{-1}(m) = K(m)$, $h(m) = h(\mu(\theta)) = F(\theta)$. It follows that for every $K(m) \in C_\mu$, $K(m) \subseteq F^{-1}(F(\theta))$. Since this is true for all $K(m) \in C_\mu$, we say that C_μ is $F - cc$.

When $\Theta = \Theta^1 \times \cdots \times \Theta^N$ and μ is decentralized, $\theta \in \mu^{-1}(m)$ if and only if $\theta^j \in \left(\mu^j\right)^{-1}(m)$ for $j = 1, , N$, where $\theta = \left(\theta^1, \ldots, \theta^N\right)$. It follows that

$$\mu^{-1}(m) = \left(\mu^1\right)^{-1}(m) \times \cdots \times \left(\mu^N\right)^{-1}(m),$$

where $\left(\mu^j\right)^{-1}(m) \subseteq \Theta^j$, $j = 1, \ldots, N$. The set $\mu^{-1}(m) = K(m)$ is a rectangle. We have argued that the set

$$C_\mu = \{K(m) : m \in M'\}$$

is a set of rectangles covering the parameter space. Moreover, every set in C_μ is $F - cc$.

We have shown that the covering induced by μ is: (1) rectangular (this follows from the decentralization) and (2) $F - cc$.

The previous shows that when a decentralized mechanism exists realizing a given goal function, then a rectangular covering of the parameter space exists, where any set of the covering is $F - cc$.

An Algorithm for Constructing a Reflexive Rectangular Method (rRM) Covering

Given a goal function $F : \Theta = \Theta^1 \times \cdots \times \Theta^N \to Z$, we construct a correspondence $V : \Theta \Rightarrow \Theta$ such that

1. for every $\theta \in \Theta$, $V(\theta)$ is rectangular (i.e., for every $\theta \in \Theta$, there are sets $K_i \subseteq \Theta^i$, $i = 1, \ldots, N$, such that $V(\theta) = K_1 \times \cdots \times K_N$),
2. V is $F - cc$ (i.e., for every $\theta \in \Theta$, $V(\theta) \subseteq F^{-1}(F(\theta))$),
3. V is *self-belonging* (i.e., for every $\theta \in \Theta$, $\theta \in V(\theta)$).
4. V is a *reflexive* correspondence (defined below). (Casually speaking, we make the rectangles as large as possible.)

Preliminaries

We assume two agents.

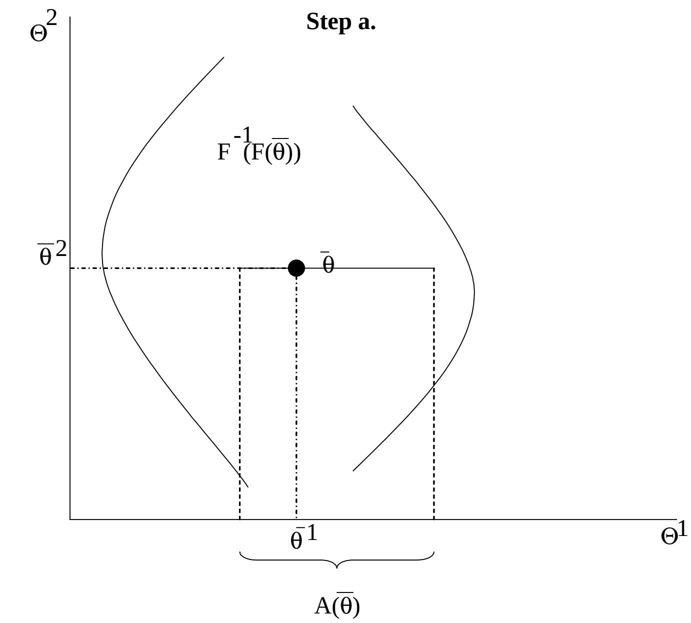
Step a. Define a correspondence $A : \Theta \Rightarrow \Theta^1$ in the following way. Given an arbitrary $\bar{\theta} = \left(\bar{\theta}^1, \bar{\theta}^2\right) \in \Theta$, choose $A\left(\bar{\theta}\right) \subseteq \Theta^1$ such that

$$\bar{\theta}^1 \in A\left(\bar{\theta}\right) \tag{2}$$

and

$$A\left(\bar{\theta}\right) \times \left\{\bar{\theta}^2\right\} \subseteq F^{-1}\left(F\left(\bar{\theta}\right)\right) \tag{3}$$

(note there may well be many ways to choose $A\left(\bar{\theta}\right)$).



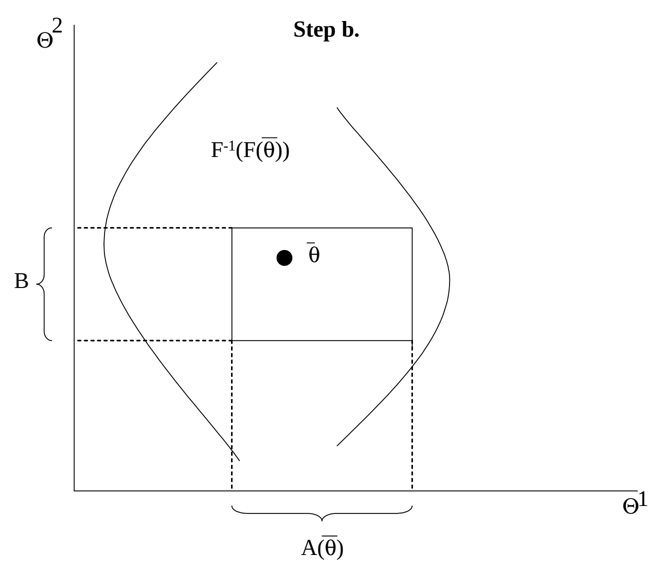
Step b. Given $A\left(\bar{\theta}\right)$, choose $B\left(A\left(\bar{\theta}\right), \bar{\theta}\right) \subseteq \Theta^2$, such that

$$\bar{\theta}^2 \in B\left(A\left(\bar{\theta}\right), \bar{\theta}\right)$$

and

$$A\left(\bar{\theta}\right) \times B\left(A\left(\bar{\theta}\right), \bar{\theta}\right) \subseteq F^{-1}\left(F\left(\bar{\theta}\right)\right).$$

Define $V\left(\bar{\theta}\right) = A\left(\bar{\theta}\right) \times B\left(A\left(\bar{\theta}\right), \bar{\theta}\right)$. Notice that is rectangular, $F - cc$, and self-belonging.



Step b′. Now we construct the rectangles to begin to illustrate how we will make them in some sense as large as possible. Rather than just following the Step b once, find all B such that

$$\bar{\theta}^2 \in B$$

and

$$A\left(\bar{\theta}\right) \times B \subseteq F^{-1}\left(F\left(\bar{\theta}\right)\right).$$

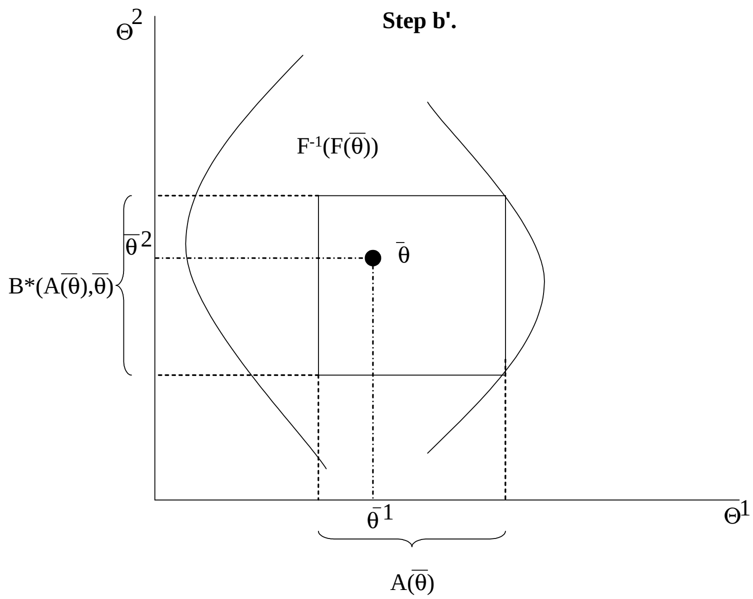
Then define

$$B^*\left(A\left(\bar{\theta}\right), \bar{\theta}\right) = \cup\left\{B \subseteq \Theta^2 : \bar{\theta}^2 \in B \text{ and } A\left(\bar{\theta}\right) \times B \subseteq F^{-1}\left(F\left(\bar{\theta}\right)\right)\right\}. \tag{4}$$

This set makes the $B-$side of the rectangle as large as possible subject to the condition that $A\left(\bar{\theta}\right) \times B^*\left(A\left(\bar{\theta}\right), \bar{\theta}\right) \subseteq F^{-1}\left(F\left(\bar{\theta}\right)\right)$. We can define the correspondence $L : \Theta \Rightarrow \Theta$ by

$$L(\theta) = A(\theta) \times B^*(A(\theta), \theta);$$

this is referred to as the *left rectangles method correspondence* (left RM correspondence). By the way in which L is defined, it is rectangular, $F - cc$, and self-belonging.



We could similarly define the right rectangles method correspondence (see your text) by starting with construction of the second agent's side of the rectangles, $B\left(\bar{\theta}\right)$. Then define

$$A^*\left(B\left(\bar{\theta}\right), \bar{\theta}\right) = \cup\left\{A \subseteq \Theta^1 : \bar{\theta}^1 \in A \text{ and } A \times B\left(\bar{\theta}\right) \subseteq F^{-1}\left(F\left(\bar{\theta}\right)\right)\right\}. \tag{5}$$

Define the *right rectangles method correspondence* (right RM correspondence) $R : \Theta \Rightarrow \Theta$ by

$$R(\theta) = A^*(B(\theta), \theta) \times B(\theta).$$

This correspondence is rectangular, $F - cc$, and self-belonging.

In general, $R(\theta) \neq L(\theta)$.

The Method

We assume two agents. We now construct a correspondence which is both a right RM and a left RM correspondence. Choose $A\left(\bar{\theta}\right) = A_1 \subseteq \Theta^1$ such that

$$\bar{\theta}^1 \in A\left(\bar{\theta}\right) = A_1$$

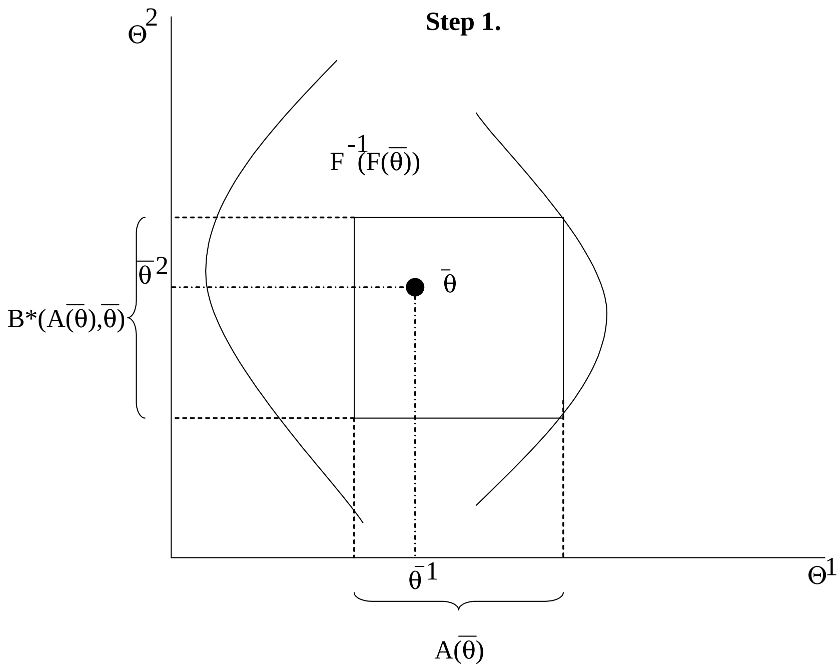
and

$$A\left(\bar{\theta}\right)\times\left\{\bar{\theta}^{2}\right\}=A_{1}\times\left\{\bar{\theta}^{2}\right\}\subseteq F^{-1}\left(F\left(\bar{\theta}\right)\right)$$

(note there may well be many ways to choose $A_1 = A\left(\bar{\theta}\right)$).

Step 1 Now find $\hat{B}\left(\bar{\theta}\right) = B_1 = B^*\left(A_1, \bar{\theta}\right)$, i.e., find

$$\hat{B}\left(\bar{\theta}\right)=B^{*}\left(A_{1},\bar{\theta}\right)=\cup\left\{B\subseteq\Theta^{2}:\bar{\theta}^{2}\in B\text{ and }A_{1}\times B\subseteq F^{-1}\left(F\left(\bar{\theta}\right)\right)\right\}.$$

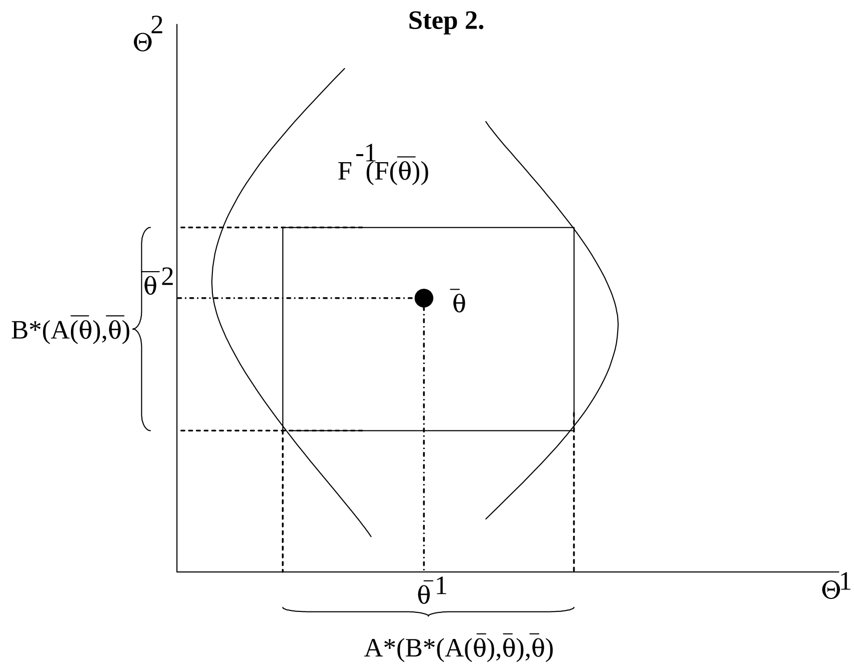


Step 2 Given $\hat{B}\left(\bar{\theta}\right) = B_1 = B^*\left(A_1, \bar{\theta}\right)$ from the previous step, determine

$$\tilde{A}\left(\bar{\theta}\right)=A_{2}=A^{*}\left(\hat{B}\left(\bar{\theta}\right),\bar{\theta}\right)=A^{*}\left(B_{1},\bar{\theta}\right); \tag{6}$$

that is, find

$$\tilde{A}\left(\bar{\theta}\right)=A^{*}\left(B_{1},\bar{\theta}\right)=\cup\left\{A\subseteq\Theta^{1}:\bar{\theta}^{1}\in A\text{ and }A\times B_{1}\subseteq F^{-1}\left(F\left(\bar{\theta}\right)\right)\right\}.$$



Step 3? Given $\tilde{A}\left(\bar{\theta}\right) = A_2 = A^*\left(B_1, \bar{\theta}\right) = A^*\left(\hat{B}\left(\bar{\theta}\right), \bar{\theta}\right)$ from the previous step, determine $\tilde{B}\left(\bar{\theta}\right) = B_2 = B^*\left(A_2, \bar{\theta}\right) = B^*\left(\tilde{A}\left(\bar{\theta}\right), \bar{\theta}\right)$. But this third step is unnecessary because by Thm 3.4.1, p. 191 (given below),

$$\underbrace{B_1 = B^*\left(A_1, \bar{\theta}\right) = \hat{B}\left(\bar{\theta}\right)}_{\text{the set that was obtained in Step 1}} = \underbrace{\tilde{B}\left(\bar{\theta}\right) = B^*\left(\tilde{A}\left(\bar{\theta}\right), \bar{\theta}\right) = B^*\left(A_2, \bar{\theta}\right) = B_2}_{\text{the set that would be obtained in Step 3}}. \tag{7}$$

This means that the algorithm ends at the end of the second step. Also, by (7), it follows that in Step 2 at (6), we can substitute $\hat{B}\left(\bar{\theta}\right) = \tilde{B}\left(\bar{\theta}\right)$ to obtain

$$\tilde{A}\left(\bar{\theta}\right) = A^*\left(\tilde{B}\left(\bar{\theta}\right), \bar{\theta}\right). \tag{8}$$

By (7) in Step 1 we found $\hat{B}\left(\bar{\theta}\right)$,which equals

$$\tilde{B}\left(\bar{\theta}\right) = B^*\left(\tilde{A}\left(\bar{\theta}\right), \bar{\theta}\right), \tag{9}$$

given the Thm which follows. Now form the rectangle as

$$V\left(\bar{\theta}\right) = \tilde{A}\left(\bar{\theta}\right) \times \tilde{B}\left(\bar{\theta}\right)$$

where (8) and (9) hold. This defines a reflexive rectangles method correspondence. This follows directly from the formal definition of such a correspondence on p. 194. Loosely speaking, this means that the "largest" side of the rectangle relating to Agent 1's characteristics is obtained given the "largest" side of the rectangle relating to Agent 2's characteristics, and that "largest" side of the rectangle relating to Agent 2's characteristics is obtained given the "largest" side of the rectangle relating to Agent 1's characteristics. The covering generated by the correspondence

$$\mathcal{C}_V = \{V(\theta) : \theta \in \Theta\}$$

is said to be reflexive. $V(\theta)$ is both a right and left RM correspondence and therefore, a *reflexive* RM correspondence, as we will argue.

Theorem 4 *Let* $\bar{\theta} \in \Theta^1 \times \Theta^2$ *and let* $A_1 \subseteq \Theta^1$*, where* $\bar{\theta}^1 \in A_1$*. If* $A_2 = A^*\left(B_1, \bar{\theta}\right)$*, then* $B^*\left(A_1, \bar{\theta}\right) = B^*\left(A_2, \bar{\theta}\right)$.

Apply the Theorem to find that

$$\tilde{B}\left(\bar{\theta}\right) = B^{*}\left(A\left(\bar{\theta}\right), \bar{\theta}\right) = B^{*}\left(\tilde{A}\left(\bar{\theta}\right), \bar{\theta}\right),$$

as stated above. We say that

$$V\left(\theta\right) = \tilde{A}\left(\theta\right) \times \tilde{B}\left(\theta\right)$$

is a *reflexive* RM correspondence since $\tilde{A}\left(\bar{\theta}\right) = A^{*}\left(\tilde{B}\left(\bar{\theta}\right), \bar{\theta}\right)$ and $\tilde{B}\left(\bar{\theta}\right) = B^{*}\left(\tilde{A}\left(\bar{\theta}\right), \bar{\theta}\right)$ (see the definition, p. 194). The covering of Θ *generated by* V is

$$\mathcal{C}_V = \left\{V\left(\theta\right) : \theta \in \Theta\right\},$$

which may be denoted by $\mathcal{C}$. The covering is said to be *reflexive.*

The method illustrated here can be extended to situations where there are N agents. The process takes N steps. See Definition 3.4.1, Theorem 3.4.2, and Theorem 3.4.3, pp. 191-6.

Constructing a Mechanism from a Covering by the Transversals Method (Section 3.5)

Math Facts, Notation

Throughout, the letter $\mathcal{C}$ is used to denote a set of sets only. The letters K and T are used to denote sets only. The set $A \backslash B$ is used to denote the set

$$\left\{a \in A : a \notin B\right\}.$$

If $f : A \to B$ is a function, then

$$f(A) = \left\{b \in B : b = f(a) \text{ for some } a \in A\right\}.$$

Definition 5 *Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of the set C. If for each $B \in \mathcal{B}$, there exists $A \in \mathcal{A}$, such that $B \subseteq A$, then $\mathcal{B}$ is a refinement of $\mathcal{A}$ (or $\mathcal{B}$ refines $\mathcal{A}$).*

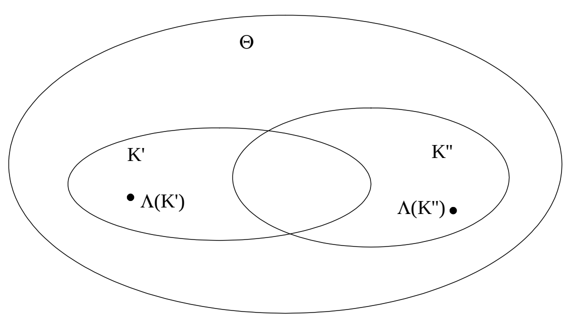
Axiom of Choice For any set A there is a function f such that for any nonempty subset B of A, $f(B) \in B$.

Constructing the Transversal

Definition 6 *A system of distinct representatives (SDR) for a covering $\mathcal{C}$ of Θ is a function $\Lambda : \mathcal{C} \to \Theta$ such that*

(SDR i) for every $K \in \mathcal{C}$, $\Lambda(K) \in K$ ($\Lambda(K)$ is said to represent K) and

(SDR ii) if $K' \neq K''$, then $\Lambda(K') \neq \Lambda(K'')$.

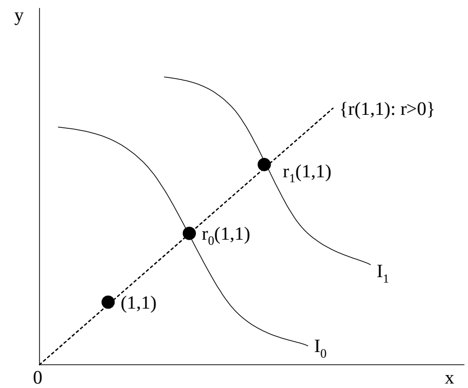


Definition 7 *Let $\mathcal{C}$ be a covering of Θ and let $\Lambda : \mathcal{C} \to \Theta$ be a system of distinct representatives for $\mathcal{C}$. The set $T = \Lambda(\mathcal{C})$ is a transversal for $\mathcal{C}$ (associated with Λ).*

Example Consider the following indifference curves over the consumption space $X = \mathbf{R}^2_+$. We assume that the consumer has a continuous weak order and prefers more to less. (See p. 47, Mas-Colell et al.) The space

$$T = \{r(1,1) : r > 0\}$$

is a transversal for the set of indifference curves (which is a covering of the consumption space X). Determining a transversal in this situation is easier than in other situations because the set of indifference curves forms a partition of the consumption set.



Proofs of the following theorem are on pp. 103 and 221.

Theorem 8 *Let $\mathcal{C}$ be an arbitrary covering of a set Θ. $\mathcal{C}$ has an SDR if and only if $\mathcal{C}$ is generated by a self-belonging correspondence $V : \Theta \rightrightarrows \Theta$.*

Proof. ($\Leftarrow$) Given a set K of the covering $\mathcal{C}_V$ generated by the self-belonging correspondence $V : \Theta \rightrightarrows \Theta$, let θ be a generator of K, i.e., $V(\theta) = K$ (defined below). Let Θ_K be the set of all generators of K, i.e.,

$$\Theta_K = \{\theta \in \Theta : V(\theta) = K\}.$$

Choose an arbitrary element $\theta_K \in \Theta_K$ as the representative of the set K (this can be done by the Axiom of Choice). Note also that if $K' \neq K''$, then $\Theta_{K'} \cap \Theta_{K''} = \varnothing$ (otherwise, there exists $\bar{\theta} \in \Theta_{K'} \cap \Theta_{K''}$, which implies that $V(\bar{\theta}) = K' = K'' = V(\bar{\theta})$, a contradiction). It follows that if $K' \neq K''$, then their representatives, $\Lambda(K') = \theta_{K'}$ and $\Lambda(K'') = \theta_{K''}$, must be different.

($\Rightarrow$) Now assume that $\mathcal{C}$ has an SDR $\Lambda : \mathcal{C} \to \Theta$. We show that there exists a self-belonging correspondence $V : \Theta \rightrightarrows \Theta$ that generates $\mathcal{C}$. Since $\mathcal{C}$ has an SDR $\Lambda : \mathcal{C} \to \Theta$, given any $K \in \mathcal{C}$, its representative is $\Lambda(K) = \theta_K \in K$. Define $V(\theta_K) = K$. V is clearly self-belonging. Since $\mathcal{C}$ is a covering of Θ, so is the set

$$\{V(\theta_K) : \theta_K \in \Lambda(\mathcal{C})\}. \square$$

Note We are given:

- a goal function $F : \Theta \to Z$
- a covering $\mathcal{C}$ of Θ that is generated by a self-belonging correspondence $V : \Theta \rightrightarrows \Theta$, i.e., $\mathcal{C}_V = \mathcal{C} = \{K : K = V(\theta) \text{ for some } \theta \in \Theta\}$ (when $K = V(\theta)$, θ is said to be a *generator* of K and may be denoted by θ_K).
- $\mathcal{C}$ is contour contained, i.e., if $K \in \mathcal{C}$, $K \subseteq F^{-1}(z)$ for some $z \in Z$ (or if $K \in \mathcal{C}$, the goal function F is constant on K)
- $\mathcal{C}$ has an SDR $\Lambda : \mathcal{C} \to \Theta$, with set $T = \Lambda(\mathcal{C})$ a transversal for $\mathcal{C}$

Lemma 9 *Given a goal function $F : \Theta \to Z$ and a transversal T,*

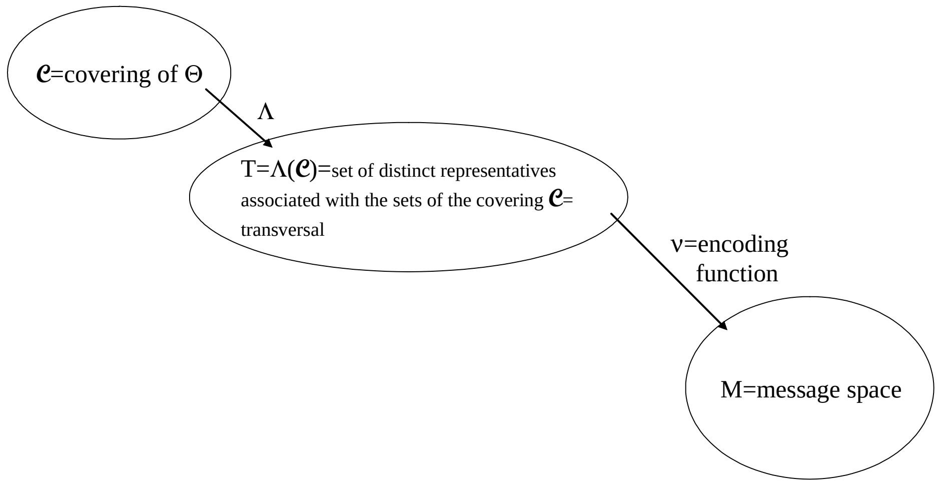
(i) if $\theta \in F^{-1}(z) \cap T$ and $\theta' \in F^{-1}(z) \cap T$, then $F(\theta) = F(\theta')$, and

(ii) if $z \neq z'$, then $\left(F^{-1}(z) \cap T\right) \cap \left(F^{-1}(z') \cap T\right) = \varnothing$.

Proof. For you. □

Let $F_T : T \to Z$ be the restriction of F to T. The partition induced by F_T on T is the same as the partition induced by F on T. The following definition is on pp. 111 and 223.

Definition 10 *Let T be a transversal and let $\nu : T \to M$ be a one-to-one function into M. Then ν is an encoding function for T.*



Note that an encoding function for T always exists. For example, define $M = T$ and $\nu(t) = t$.

The function $\nu : T \to M$ *encodes* the points of the transversal in (perhaps) fewer variables than the number of parameters in Θ. More formally, write $\nu(T) = M' \subseteq M$ and $\nu^{-1} : \nu(T) \Rightarrow T$. The function induces the partition

$$\mathcal{P}(T, \nu) = \left\{ K \subseteq T : K \subseteq \nu^{-1}(m) \text{ for some } m \in M' \right\}.$$

Throughout, ν is used to denote an encoding function only.

Example Consider the previous example. For $r(1,1) \in T$, define $\nu : T \to \mathbf{R}$ by

$$\nu\left(r(1,1)\right) = r.$$

You should argue that ν is one-to-one. By using the encoding function, we have represented the information in T, a space of two dimensions, in a space of one dimension.

Lemma 11 *The collection $\mathcal{P}(T, \nu)$ is $F - cc$ if and only if $\mathcal{P}(T, \nu)$ is a refinement of $\mathcal{P}(T, F) = \mathcal{P}(T, F_T)$.*

Proof. ($\Rightarrow$) Suppose that $\mathcal{P}(T, \nu)$ is $F - cc$. We need to show that $\mathcal{P}(T, \nu)$ is a refinement of $\mathcal{P}(T, F)$, i.e., for every $K \in \mathcal{P}(T, \nu)$ there exists $K' \in \mathcal{P}(T, F)$ such that $K \subseteq K'$. Let $K \in \mathcal{P}(T, \nu)$ and let $\theta \in K$. Let $z = F(\theta)$. Then there exists $K' \in \mathcal{P}(T, F)$ such that $K' = F^{-1}(z)$. Because $\mathcal{P}(T, \nu)$ is $F - cc$, if $\theta' \in K$, then $\theta' \in F^{-1}(z) = K'$. This shows that $K \subseteq K'$.

($\Leftarrow$) For you. □

Corollary 12 *If ν^{-1} is singleton-valued (that is, ν^{-1} is a function), then $\mathcal{P}(T, \nu)$ is $F - cc$.*

Proof. For you. □

Definition 13 *A function $f : T \to M$ is $F-$compatible if $\mathcal{P}(T, f)$ is $F - cc$.*

References

Hurwicz and Reiter.
Munkres, *Topology.*
Suppes, *Axiomatic Set Theory.*