ECN 612 - HANDOUT - FALL 2009
Chapter 1: Introduction to Mechanism Design

NOTATION, DEFINITIONS

Θ is the environment space
Z is the outcome space
M is the message space; contains messages available for communication
$F : \Theta \to Z$ is the goal function
$\mu : \Theta \Rightarrow M$ is the (equilibrium) message correspondence
$h : M \to Z$ is the outcome function

The triple $\pi = (M, \mu, h)$ is a *mechanism.* The mechanism when operated in the environment θ yields outcomes $h(\mu(\theta))$ in Z. π *realizes* F (on Θ) if for all $\theta \in \Theta$, $h(\mu(\theta)) = F(\theta)$. This idea is represented in the following diagram.

$$\begin{array}{ccccc} \Theta & & \overset{F}{\to} & & Z \\ & \underset{\mu}{\searrow} & & \underset{h}{\nearrow} & \\ & & M & & \end{array}$$

Notes It could be the case that $\Theta = \prod_{i=1}^{N} \Theta^i$. It could be the case that $M = \prod_{i=1}^{N} M^i$. Finally, it could be the case that $\mu = \prod_{i=1}^{N} \mu^i$.

Notes Mechanisms may or may not take account of the individuals' strategic considerations. For example, neither the competitive mechanism nor the Lindahl mechanisms account for strategic action on the part of individuals. However, the following mechanisms do: Clarke, Groves, Groves-Ledyard, and Cournot-Lindahl.

Notes The environment space, for example, could be the set of all convex preference relations. It could be a set of quasilinear utility functions. It could be a set of quasilinear quadratic utility functions.

The equilibrium message space is determined by the agents' behaviors. For example, it could be amounts demanded of the goods at a competitive equilibrium. The equilibrium message space could be the amounts declared under the Groves-Ledyard or Cournot-Lindahl mechanisms at a Nash equilibrium. It could be the preferences declared at the dominant strategy equilibrium under the Groves mechanism.

The goal function F could be the Pareto correspondence. It could be the Walrasian outcome correspondence. It could be the correspondence yielding outcomes that are both Pareto efficient and individually rational. It could be a correspondence yielding outcomes that are envy free. It could be a correspondence yielding fair outcomes.

AN EXAMPLE OF MECHANISM DESIGN: A WALRASIAN EXAMPLE

Consider a two person, two good economy, where individuals' consumption sets are contained in $\mathbf{R}^2_+$. Assume that

$$U^i(X_i, Y_i) = \alpha_i X_i - \tfrac{1}{2}\beta_i X_i^2 + Y_i,$$

where $\alpha_i > 0$, $\beta_i > 0$, $i = 1, 2$. Let (w_i, v_i) be initial endowments of the commodities, $i = 1, 2$. Assume these are restricted in the example so that the utility functions are strictly increasing with respect to X_i, $i = 1, 2$. Let

$$x_i = X_i - w_i.$$

denote net trades in X by individual i, $i = 1, 2$. The utility functions become

$$\tilde{u}^i(x_i, Y_i) = \alpha_i(x_i + w_i) - \tfrac{1}{2}\beta_i(x_i + w_i)^2 + Y_i,$$

$i = 1, 2$.

The goal function F_W is to achieve the Walrasian trade in each environment given by the parameters. It follows that at the outcome,

$$x_1 + x_2 = 0,\ y_1 + y_2 = 0.$$

Assume that initial endowments are constant from this point. Set $\gamma_i = \alpha_i - \beta_i w_i$, $i = 1, 2$. Set $\theta^1 = (a_1, a_2) = (\beta_1, \gamma_1) = (\beta_1, \alpha_1 - \beta_1 w_1)$ and $\theta^2 = (b_1, b_2) = (\beta_2, \gamma_2) = (\beta_2, \alpha_2 - \beta_2 w_2)$. Outcomes are net trades and we focus attention on x_2. Outcome space Z is $\mathbf{R}$.

The Walrasian function F_W associates with each $\theta = (\theta^1, \theta^2) \in (\Theta^1, \Theta^2)$ its unique Walrasian trade. It can be shown that

$$F_W(\theta) = \frac{b_2 - a_2}{b_1 + a_1}. \tag{1}$$

To see this, we may assume that

$$u^i(x_i, Y_i) = (\alpha_i - \beta_i w_i)x_i - \tfrac{1}{2}\beta_i x_i^2 + Y_i + k,$$

$i = 1, 2$, or, in alternative notation,

$$u^1(x_1, Y_1) = a_2 x_1 - \tfrac{1}{2}a_1 x_1^2 + Y_1,$$

$$u^2(x_2, Y_2) = b_2 x_2 - \tfrac{1}{2}b_1 x_2^2 + Y_2.$$

At the Walrasian outcome, given the individuals' constrained utility maximization and letting Y be the numeraire good,

$$\begin{array}{rcl} -a_1 x_1 + a_2 & = & p \\ -b_1 x_2 + b_2 & = & p. \end{array} \tag{2}$$

Given the market clearing condition,

$$x_1 + x_2 = 0. \tag{3}$$

From (2) and (3),

$$\begin{array}{rcl} a_1 x_2 + a_2 & = & p \\ -b_1 x_2 + b_2 & = & p. \end{array} \tag{4}$$

From these we find that

$$\begin{array}{rcl} x_2 & = & \dfrac{b_2 - a_2}{b_1 + a_1} \\ p & = & \dfrac{a_1 b_2 + a_2 b_1}{b_1 + a_1}. \end{array} \tag{5}$$

In this example, the goal function F_W is the Walrasian one. It maps agents' characteristics onto Walrasian outcomes (obtained by consumers maximizing utility given their budget constraints and so that markets clear).

In this case,

$$\Theta = \Theta^1 \times \Theta^2 = \left\{(a_1, a_2, b_1, b_2) \in \mathbf{R}^4 :\ a_i > 0,\ b_i > 0,\ i = 1, 2\right\}.$$

(To see how we arrive at the restrictions on the $\theta^1 = (a_1, a_2)$, $\theta^2 = (b_1, b_2)$, recall that the consumers' original utility functions (with domains contained in $\mathbf{R}^2_+$) are

$$U^i(X_i, Y_i) = \alpha_i X_i - \tfrac{1}{2}\beta_i X_i^2 + Y_i,$$

$i = 1, 2$, $\theta^1 = (a_1, a_2) = (\beta_1, \gamma_1) = (\beta_1, \alpha_1 - \beta_1 w_1)$, and $\theta^2 = (b_1, b_2) = (\beta_2, \gamma_2) = (\beta_2, \alpha_2 - \beta_2 w_2)$. To ensure that the utility functions are strictly increasing and strictly quasiconcave, we restrict the parameters and domains of the utility functions by the following: $\alpha_i > 0$, $\beta_i > 0$, and $X_i < \dfrac{\alpha_i}{\beta_i}$. To see this, note that

$$U^i_{XX}(X_i, Y_i) = -\beta_i < 0 \Leftrightarrow \beta_i > 0 \tag{6}$$

and that

$$U_X^i\left(X_i, Y_i\right) = \alpha_i - \beta_i X_i > 0 \Leftrightarrow X_i < \frac{\alpha_i}{\beta_i}, \tag{7}$$

where we have used (3). Given (3), (2), and the fact that we want the domain of the utility function to contain positive amounts of X, it follows that we want to have $\alpha_i > 0$. Given the restriction at (2) and the fact that the consumption sets should contain the initial endowments, we also want the following restriction:

$$w_i < \frac{\alpha_i}{\beta_i}. \tag{8}$$

Now note that $a_1 = \beta_1 > 0$, $b_1 = \beta_2 > 0$ by (6). From (7) and (8), it follows that $a_2 = \alpha_1 - \beta_1 w_1 > 0$, $b_2 = \alpha_2 - \beta_2 w_2 > 0$.)

At the end of the previous handout, we found that $Z = \mathbf{R}$ and that the Walrasian goal function is

$$F_W(\theta) = F_W(a_1, a_2, b_1, b_2) = \frac{b_2 - a_2}{b_1 + a_1}.$$

The latter represents the net trade in the second good for the second individual at the Walrasian outcome (how would you determine the other net trades?). We have clearly indicated Θ, F_W, and Z in the top part of the following diagram.

$$\begin{array}{ccccc} \Theta & & \overset{F_W}{\longrightarrow} & & Z \\ & \searrow & & \nearrow & \\ & \mu\ ? & & h\ ? & \\ & & M\ ? & & \end{array}$$

In the next two sections, we fill in the question marks in the diagram above in two different ways; that is, we determine two mechanisms which realize the Walrasian goal function.

THE COMPETITIVE MECHANISM

We assume that consumers follow constrained utility maximization and that markets clear. But we need to determine μ_c, M_c, and h_c so that what we think of as the competitive mechanism actually qualifies as a mechanism. From (4), we find that

$$\begin{array}{rcl} a_2 + a_1 x - p & = & 0 \\ b_2 - b_1 x - p & = & 0, \end{array} \tag{9}$$

where we have rearranged the equations and set $x_2 = x$. Messages in the competitive mechanism will be of the form $m = (m_1, m_2) = (x, p)$. The message space is $M_c \subseteq \mathbf{R} \times \mathbf{R}_{++}$. Use (9) to find that the individual message correspondences are

$$\begin{array}{rcl} \mu_c^1(a_1, a_2) & = & \left\{(m_1, m_2) \in M_c : a_2 + a_1 m_1 - m_2 = 0\right\}, \\ \mu_c^2(b_1, b_2) & = & \left\{(m_1, m_2) \in M_c : b_2 - b_1 m_1 - m_2 = 0\right\}. \end{array} \tag{10}$$

The (group) equilibrium message correspondence is obtained by taking the intersection of the individuals' message correspondences. We find that

$$\mu_c(a_1, a_2, b_1, b_2) = \left\{(m_1, m_2) \in M_c : m_1 = \frac{b_2 - a_2}{b_1 + a_1},\ m_2 = \frac{a_1 b_2 + a_2 b_1}{b_1 + a_1}\right\}. \tag{11}$$

Define the outcome function $h_c : M_c \to Z$ as the projection onto its first argument, i.e.,

$$h_c(m_1, m_2) = m_1 = \frac{b_2 - a_2}{b_1 + a_1}.$$

Now notice that

$$h_c(\mu_c(a_1, a_2, b_1, b_2)) = h_c\left(\frac{b_2 - a_2}{b_1 + a_1}, \frac{a_1 b_2 + a_2 b_1}{b_1 + a_1}\right) = \frac{b_2 - a_2}{b_1 + a_1} = F_W(a_1, a_2, b_1, b_2).$$

We have shown that the triple $\pi_c = (M_c, \mu_c, h_c)$ defined in this section is a mechanism, the competitive mechanism, and it realizes the Walrasian goal function.

Privacy Preservation in the Competitive Mechanism

Let

$$F_W^{-1}(c) = \left\{ \left(\theta^1, \theta^2\right) \in \Theta^1 \times \Theta^2 : \; c = \frac{b_2 - a_2}{b_1 + a_1} \right\}; \tag{12}$$

this is the set of points in the environment space which map to c (in the outcome space Z), the net trade in the second good for the second individual. If $c \in Z$, but c is not the image of something in Θ under F_W, then $F_W^{-1}(c) = \varnothing$. (There is an analogy between this type of set and an indifference curve. Given the equation of an indifference curve $U(x, y) = k$, the set $U^{-1}(k) = \{(x, y) \in X : U(x, y) = k\}$ is the indifference set associated with the utility level k. It is a level set associated with the utility function.)

$$\Theta \quad \underset{\overset{F_W^{-1}}{\longleftarrow}}{\overset{F_W}{\longrightarrow}} \quad Z$$

Notice that $F_W^{-1}(c)$ at (12) is equal to the set of points $\left(\theta^1, \theta^2\right) = (a_1, a_2, b_1, b_2)$ such that

$$a_2 + ca_1 = d \tag{13}$$

and

$$b_2 - cb_1 = d \tag{14}$$

for some $d \in \mathbf{R}$. (It may be the case that such a point (a_1, a_2, b_1, b_2) does not exist.) Let

$$\left\{(a_1, a_2, b_1, b_2) \in \Theta^1 \times \Theta^2 : \; a_2 + ca_1 = d \text{ and } b_2 - cb_1 = d\right\}; \tag{15}$$

this set is referred to as a *rectangle* since it is the Cartesian product of two sets, one of which is in the parameter space of Individual 1 and the other in the parameter space of Individual 2. That is, (15) can be rewritten as

$$\left\{(a_1, a_2) \in \Theta^1 : \; a_2 + ca_1 = d\right\} \times \left\{(b_1, b_2) \in \Theta^2 : \; b_2 - cb_1 = d\right\}; \tag{16}$$

Notice that this rectangle is associated with the two real numbers c and d; denote this rectangle by $R(c, d)$.

The idea that $F_W^{-1}(c)$ at (12) is equal to the set of points $\left(\theta^1, \theta^2\right) = (a_1, a_2, b_1, b_2)$ such that (13) and (14) hold for some $d \in \mathbf{R}$ can be expressed as

$$F_W^{-1}(c) = \bigcup_{d \in \mathbf{R}} \left\{(a_1, a_2, b_1, b_2) \in \Theta^1 \times \Theta^2 : \; a_2 + ca_1 = d \text{ and } b_2 - cb_1 = d\right\}. \tag{17}$$

Use (16) to rewrite (**??**) as

$$F_W^{-1}(c) = \bigcup_{d \in \mathbf{R}} \left(\left\{(a_1, a_2) \in \Theta^1 : \; a_2 + ca_1 = d\right\} \times \left\{(b_1, b_2) \in \Theta^2 : \; b_2 - cb_1 = d\right\}\right). \tag{18}$$

Given (9) and (10) on the previous handout, we see that

$$m_1 = x = c, \; m_2 = p = d.$$

Notice that

$$\left\{\left\{(a_1, a_2) \in \Theta^1 : a_2 + ca_1 = d\right\} \times \left\{(b_1, b_2) \in \Theta^2 : b_2 - cb_1 = d\right\} : \; d \in \mathbf{R}\right\} \tag{19}$$

is a partition of $F_W^{-1}(c)$ (this follows since any two distinct nonempty sets are disjoint (i.e., they have an empty intersection) and the union of all the sets is $F_W^{-1}(c)$). It follows that $\{R(c, d) : \; c \in \mathbf{R}, \, d \in \mathbf{R}\}$ is a partition of $\Theta = \Theta^1 \times \Theta^2$. We say that the partition is an $F - cc$ ($F-$contour contained) covering. (This is analogous to the fact that the set of all indifference sets forms a partition of an individual's consumption set.)

Notice that the $F - cc$ covering has the following informational properties:

(i) Each agent can verify the joint message $(m_1, m_2) = (x, p) = (c, d)$ knowing only his/her own parameters (not knowing the other's parameters). This is the *privacy-preserving* property of the covering and of the mechanism.

(ii) Given their parameters, the agents independently and simultaneously verify a particular message $(m_1, m_2) = (x, p) = (c, d)$ if and only if $m_1 = x = c = F(a_1, a_2, b_1, b_2)$ and $m_2 = p = d$.

(iii) The elements of the message space consist of two real numbers, i.e., the dimension of the message space is 2, although the number of parameters characterizing the agents' preferences is 4.

In this section, we took the competitive mechanism and constructed an $F-$covering (which in this case was a partition) of the level sets of F_W by rectangles.

DERIVING A MECHANISM (NOT THE COMPETITIVE MECHANISM) FROM A COVERING FOR THE WALRASIAN GOAL FUNCTION

On the other hand, in this section, we illustrate that if we find an F_W-covering of the level sets and we can find a way to label the rectangles by $m_1, \ldots, m_r$, then we can obtain a system of equilibrium messages defining a mechanism which satisfies (i) and (ii) above. In this way, we define another mechanism realizing F_W; it is called the *parameter transfer mechanism from* 1 *to* 2 (abbreviated $PT_{1\to 2}$ or PT).

We construct the rectangles on $\Theta^1 \times \Theta^2$. Let

$$A(d_1, d_2) = \left\{(a_1, a_2) \in \Theta^1 : \ a_1 = d_1, \ a_2 = d_2\right\} = \{(d_1, d_2)\}$$

and

$$B(c) = \left\{(b_1, b_2) \in \Theta^2 : \ \frac{b_2 - d_2}{b_1 + d_1} = c\right\}.$$

Then a representative rectangle is

$$A(d_1, d_2) \times B(c).$$

I will leave it for you to verify that if $(d_1, d_2, c) \neq (d_1', d_2', c')$, then

$$(A(d_1, d_2) \times B(c)) \cap (A(d_1', d_2') \times B(c')) = \varnothing.$$

Furthermore,

$$\bigcup_{d_1, d_2, c \in \mathbf{R}} (A(d_1, d_2) \times B(c)) = F_W^{-1}(c),$$

as you should show. We have argued that

$$\{A(d_1, d_2) \times B(c) : \ d_1, d_2, c \in \mathbf{R}\}$$

is a partition of $F_W^{-1}(c)$. Now we take the labeling of the rectangles to construct the equilibrium message correspondence. Let

$$m_1^1 = d_1, \ m_2^1 = d_2, \ m^2 = c.$$

And let

$$\begin{aligned} \mu_{PT}^1(a_1, a_2) &= \left\{\left(m_1^1, m_2^1, m^2\right) \in M_{PT} : \ m_1^1 = a_1, \ m_2^1 = a_2\right\}, \\ \mu_{PT}^2(b_1, b_2) &= \left\{\left(m_1^1, m_2^1, m^2\right) \in M_{PT} : \ \frac{b_2 - m_2^1}{b_1 + \ m_1^1} = m^2\right\}. \end{aligned} \tag{20}$$

(These individual message correspondences may be rewritten in equation form. In each case, we seek the $m = \left(m_1^1, m_2^1, m^2\right)$ such that the equation holds.

$$g_1^1(m, a) = m_1^1 - a_1 = 0$$

$$g_2^1(m, a) = m_2^1 - a_2 = 0$$

$$g^2(m, b) = \frac{b_2 - m_2^1}{b_1 + \ m_1^1} - m^2 = 0)$$

The image of a under μ_{PT}^1 would be the $m = \left(m_1^1, m_2^1, m^2\right)$ such that the first two equations hold, the image of b under μ_{PT}^2 would be the $m = \left(m_1^1, m_2^1, m^2\right)$ such that the last equation holds. The image of (a, b) under μ_{PT}

would be the $m = \left(m_1^1, m_2^1, m^2\right)$ such that all three equations hold. Note that $\mu_{PT}(a,b) = \mu_{PT}^1(a) \cap \mu_{PT}^2(b)$; it follows that the equilibrium message correspondence is privacy-preserving. Define the outcome function as

$$h_{PT}\left(m_1^1, m_2^1, m^2\right) = m^2.$$

The mechanism $\pi_{PT} = (\mu_{PT}, M_{PT}, h_{PT})$ realizes F_W. Its dimension is 3.

Properties of this Mechanism

(i) Each agent can verify the joint message $\left(m_1^1, m_2^1, m^2\right)$ knowing only his/her own parameters (not knowing the other's parameters). This is the *privacy-preserving* property of the covering and of the mechanism.

(ii) Given their parameters, the agents independently and simultaneously verify a particular message $\left(m_1^1, m_2^1, m^2\right)$ if and only if $m_1^1 = a_1$, $m_2^1 = a_2$ and $m_2 = x = F_W(a_1, a_2, b_1, b_2)$.

(iii) The elements of the message space consist of 3 real numbers (i.e., they are vectors with three components), i.e., the dimension of the message space is 3, although the number of parameters characterizing the agents' preferences is 4.

INFORMATIONAL PROPERTIES OF THE TWO MECHANISMS

Comparisons/Contrasts

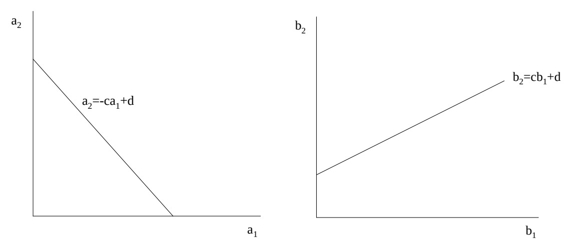
1. The competitive mechanism uses two dimensional messages (m_1, m_2), the parameter transfer uses three dimensional messages $\left(m_1^1, m_2^1, m^2\right)$.

2. The size of the messages of the competitive mechanism would be the same even if the number of parameters describing the individuals' preferences increased. The size of the parameter transfer messages would increase.

3. The rectangles of the $F_W - cc$ covering under the competitive mechanism are of the form:

$$\left\{(a_1, a_2) \in \Theta^1 : \; a_2 + ca_1 = d\right\} \times \left\{(b_1, b_2) \in \Theta^2 : \; b_2 - cb_1 = d\right\};$$

this is a two dimensional rectangle in $\mathbf{R}^4$.

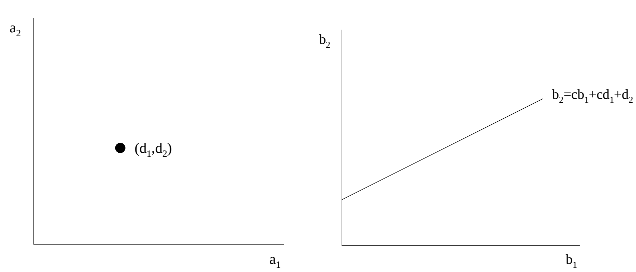


The rectangles of the $F_W - cc$ covering under the competitive mechanism are of the form:

$$\left\{(a_1, a_2) \in \Theta^1 : \; a_1 = d_1, \; a_2 = d_2\right\} \times \left\{(b_1, b_2) \in \Theta^2 : \; \frac{b_2 - d_2}{b_1 + d_1} = c\right\} =$$

$$\{(d_1, d_2)\} \times \left\{(b_1, b_2) \in \Theta^2 : \; \frac{b_2 - d_2}{b_1 + d_1} = c\right\};$$

this is a one dimensional rectangle in $\mathbf{R}^4$.



4. Under the competitive mechanism each individual has one equation to verify.

$$\text{Individual 1} \quad a_2 + m_1 a_1 - m_2 = 0$$

$$\text{Individual 2} \quad b_2 - m_1 b_1 - m_2 = 0$$

   Under the parameter transfer mechanism, the first individual has two equations to verify, the second has one equation to verify.

$$\text{Individual 1} \quad m_1^1 = a_1,\ m_2^1 = a_2$$

$$\text{Individual 2} \quad \frac{b_2 - m_2^1}{b_1 + \ m_1^1} = m^2$$

The Rectangles Method Applied to the Walrasian Goal Function - Informal

In each of the previous two examples, given the environment space Θ, the goal function F_W, and the outcome space Z, we designed a mechanism by:

1. constructing a rectangular covering (an $F_W - cc$ covering) of each level set $F_W^{-1}(c)$.
2. labeling the rectangles in the covering, producing a *product structure* for the goal function.
3. using the labels on the rectangles to create a message space, equilibrium equations, and the outcome function.

In this section, we present an algorithm describing this method. We are given the goal function $F : \Theta^1 \times \Theta^2 \rightarrow Z$.

Outline of the Rectangular Method (Assume Two Individuals)

Constructing the Rectangles

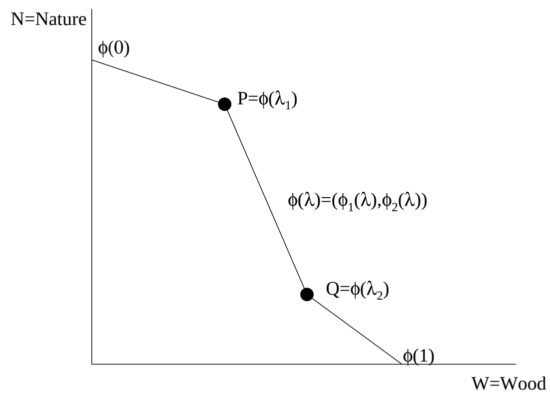
1. select a base point $\bar{\theta} \in \Theta^1 \times \Theta^2$.
2. to start the construction of a rectangle, select a starting agent, say agent 1, and then select a θ^1 side for the rectangle, denoted by $A\left(\bar{\theta}\right)$, so that the rectangle $A\left(\bar{\theta}\right) \times \bar{\theta}^2 \subseteq F^{-1}\left(F\left(\bar{\theta}\right)\right)$ (this means the rectangle is contained in the level set associated with the outcome to which $\bar{\theta}$ is assigned).
3. now construct the θ^2 side for the rectangle, denoted by $B^*\left(A\left(\bar{\theta}\right), \bar{\theta}\right)$, so that the set is the largest possible subject to the constraint that the rectangle $A\left(\bar{\theta}\right) \times B^*\left(A\left(\bar{\theta}\right), \bar{\theta}\right) \subseteq F^{-1}\left(F\left(\bar{\theta}\right)\right)$ (the latter set is the level set of which $\bar{\theta}$ is a member).

EXAMPLE: A NATIONAL FOREST

Let $\phi : [0,1] \rightarrow \mathbf{R}_+^2$ be a parametric equation describing the levels of wood and nature produced for varying intensity of logging in a forest, i.e.,

$$\phi\left(\lambda\right) = \left(\phi_1\left(\lambda\right), \phi_2\left(\lambda\right)\right),$$

where $\phi_1\left(\lambda\right)$ is the amount of wood (W) produced and $\phi_2\left(\lambda\right)$ is the level of nature (N) preserved when the intensity of logging is λ. We assume that the technology is represented in the following diagram. Note that as λ increases, W increases and N decreases. There are kinks at $\lambda = \lambda_1$ and $\lambda = \lambda_2$.



Assume that Agent 1 represents the loggers and Agent 2 represents the preservationists. Each has an associated political action function $P_i : [0,1] \times \Theta^i \to \mathbf{R}$, $i = 1, 2$ (shown in the graph below). These represent the levels of political pressure applied by these agents, which depend on their characteristics and the intensity level of logging. More precisely,

$$P_1(\lambda, a) = P_1(\lambda, a_1, a_2), \;\; P_2(\lambda, b) = P_2(\lambda, b_1, b_2),$$

where $a = (a_1, a_2)$ is the vector of characteristics of Agent 1 and $b = (b_1, b_2)$ is the vector of characteristics of Agent 2. We assume that P_1 is strictly decreasing with respect to λ (as the logging intensity increases, the political pressure applied by the loggers decreases), and P_2 is strictly increasing with respect to λ (as the logging intensity increases, the political pressure applied by the preservationists increases). Also,

$$P_1(0, a) = \tau^1_{\max}, \;\; P_1(1, a) = \tau^1_{\min}, \; P_2(0, b) = \tau^2_{\min}, \;\; P_2(1, b) = \tau^2_{\max}$$

for all $a \in \Theta^1$ and all $b \in \Theta^2$. The graphs of both P_1 and P_2 (as functions of λ) are made up of line segments. More specifically, the graph of $P_1(\lambda, a_1, a_2)$ (as a function of λ) is made up of line segments between the following pairs of points:

$$\left\{ \left(\left(0, \tau^1_{\max}\right), (\lambda_1, a_1) \right), \left((\lambda_1, a_1), (\lambda_2, a_2) \right), \left((\lambda_2, a_2), \left(0, \tau^1_{\min}\right) \right) \right\};$$

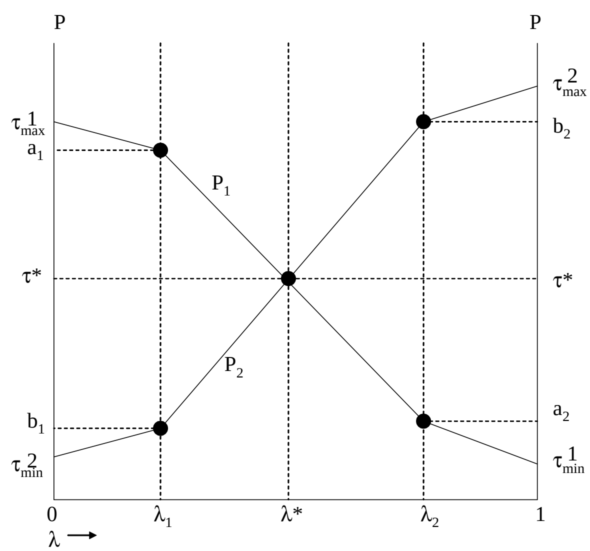
similarly, $P_2(\lambda, b_1, b_2)$ (as a function of λ) is made up of line segments between the following pairs of points:

$$\left\{ \left(\left(0, \tau^2_{\min}\right), (\lambda_1, b_1) \right), \left((\lambda_1, b_1), (\lambda_2, b_2) \right), \left((\lambda_2, b_2), \left(0, \tau^2_{\max}\right) \right) \right\}.$$

It is assumed throughout that

$$\Theta^1 = \left\{ (a_1, a_2) : \tau^1_{\min} < a_2 < a_1 < \tau^1_{\max} \right\}, \; \Theta^2 = \left\{ (b_1, b_2) : \tau^2_{\min} < b_1 < b_2 < \tau^2_{\max} \right\}.$$

The political pressure functions are represented in the following diagram.



We assume that the Forester chooses that level of intensity at which the political pressure applied by each agent is the same. That is, the goal function is

$$F_{NF}(a_1, a_2, b_1, b_2) = \lambda^* \tag{21}$$

(see the diagram above). However, the Forester does not know the characteristics of the agents, and, therefore, cannot determine directly λ^*, the intensity level of logging at which political pressures are equal. We derive a few mechanisms which realize this goal function.

(COMPLETE) REVELATION MECHANISM

$$\begin{aligned}
\mu_R^1(a_1, a_2) &= \left\{\left(m_1^1, m_2^1, m_1^2, m_2^2\right) \in M_R : m_1^1 = a_1,\ m_2^1 = a_2\right\}, \\
\mu_R^2(b_1, b_2) &= \left\{\left(m_1^1, m_2^1, m_1^2, m_2^2\right) \in M_R : m_1^2 = b_1,\ m_2^2 = b_2\right\}.
\end{aligned}$$

Define

$$\mu_R(a_1, a_2, b_1, b_2) = \mu_R^1(a_1, a_2) \cap \mu_R^2(b_1, b_2);$$

notice that this is privacy-preserving. Note that $M_R = \Theta^1 \times \Theta^2$. For $m = \left(m_1^1, m_2^1, m_1^2, m_2^2\right) \in \mu_R(a_1, a_2, b_1, b_2)$, define $h_R(m) = h_R(a_1, a_2, b_1, b_2) = \lambda^*$. By the way in which the mechanism $\pi_R = (\mu_R, M_R, h_R)$ was constructed, it realizes the goal function at (21). The message space is of dimension four.

PARAMETER TRANSFER MECHANISM

I will leave this for you (see p. 57). The message space is of dimension three.

A MECHANISM WITH THE SMALLEST POSSIBLE MESSAGE SPACE, A TWO DIMENSIONAL ONE

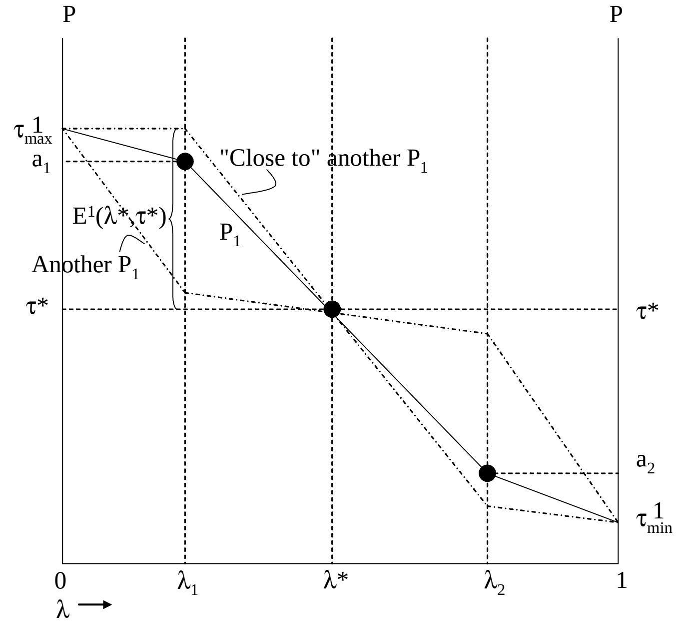
(I leave for you to argue that a mechanism with a one dimensional message space doesn't exist.) We construct a mechanism with a two dimensional space using the rectangles method. This method is described on p. 45 of your text and earlier in this handout):

1. select a base point $\bar{\theta} \in \Theta^1 \times \Theta^2$.

2. to start the construction of a rectangle, select a starting agent, say agent 1, and then select a θ^1 side for the rectangle, denoted by $A(\bar{\theta})$, so that the rectangle $A(\bar{\theta}) \times \bar{\theta}^2 \subseteq F^{-1}\left(F\left(\bar{\theta}\right)\right)$ (this means the rectangle is contained in the level set associated with the outcome to which $\bar{\theta}$ is assigned).

3. now construct the θ^2 side for the rectangle, denoted by $B^*\left(A\left(\bar{\theta}\right), \bar{\theta}\right)$, so that the set is the largest possible subject to the constraint that the rectangle $A\left(\bar{\theta}\right) \times B^*\left(A\left(\bar{\theta}\right), \bar{\theta}\right) \subseteq F^{-1}\left(F\left(\bar{\theta}\right)\right)$ (the latter set is the level set of which $\bar{\theta}$ is a member).

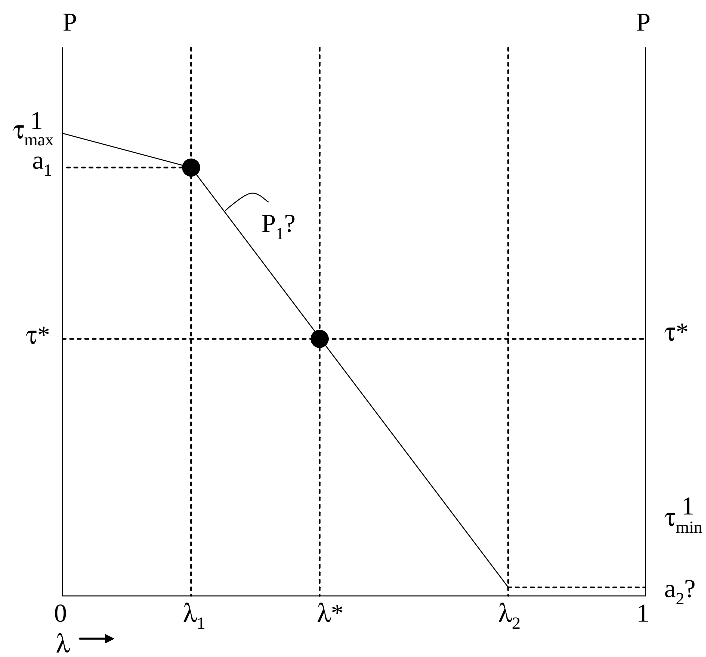
In this particular case, we select the base point (a_1, a_2, b_1, b_2) (illustrated in the previous diagram) and note that $F^{-1}(F(a_1, a_2, b_1, b_2)) = F^{-1}(\lambda^*)$. Now we want to determine rectangles contained in $F^{-1}(\lambda^*)$ which are in some sense the "largest" possible. We illustrate how to do this. Notice that there is a lot of leeway with respect to what P_1 and P_2 functions will ultimately yield an outcome of λ^*. First, assume that τ is fixed at τ^*. Then we can change (a_1, a_2, b_1, b_2) and their associated P_1 and P_2 functions (so that these functions satisfy their required properties) and still have them intersect at (λ^*, τ^*). In the following diagram, we illustrate how we can alter the P_1 function and still retain the intersection at (λ^*, τ^*).



Notice that if we set a_1 and constrain P_1 to satisfy its required properties (and go through the point (λ^*, τ^*)), then a_2 is determined. More precisely, it can be shown that

$$a_2 = \frac{\tau^* - \mu a_1}{1-\mu}, \text{ where } \mu = \frac{\lambda - \lambda_1}{\lambda_1 - \lambda_2} \tag{22}$$

(see p. 59 of your text), where we need to place appropriate restrictions. We cannot choose $a_1 < \tau^1_{\max}$ to be too large because that would mean that a_2 would be too small (the constraint that $\tau^1_{\min} < a_2$ would be violated). This is illustrated in the diagram below.



Summarizing the previous paragraph, we write

$$a_2 = \xi_1 \left(a_1, (\lambda^*, \tau^*)\right),$$

where this function is defined on the set

$$D_1 (\lambda^*, \tau^*) = \left\{a_1 : a_1 < \tau^1_{\max} \text{ and } a_2 = \xi_1 \left(a_1, (\lambda^*, \tau^*)\right) > \tau^1_{\min}\right\}.$$

We find that the set

$$E^1(\lambda^*, \tau^*) = \left\{(a_1, a_2) \in \Theta^1 : (a_1, a_2) = (a_1, \xi_1(a_1, (\lambda^*, \tau^*))),\ \text{where } a_1 \in D_1(\lambda^*, \tau^*)\right\}$$

is the largest set contained in Θ^1 such that the P_1 function goes through the point (λ^*, τ^*).

We can similarly change the P_2 function and still retain the intersection at (λ^*, τ^*). I will leave you to see the analogous development on p. 61 of your text. The set

$$E^2(\lambda^*, \tau^*) = \left\{(b_1, b_2) \in \Theta^2 : (b_1, b_2) = (b_1, \xi_2(b_1, (\lambda^*, \tau^*))),\ \text{where } b_1 \in D_2(\lambda^*, \tau^*)\right\}$$

is defined in a way analogous to the set $E^1(\lambda^*, \tau^*)$.

Notice that the set $E^1(\lambda^*, \tau) \times E^2(\lambda^*, \tau)$:

1. is a rectangular subset of the parameter space Θ,
2. is contained in the level set $F_{NF}^{-1}(\lambda^*)$,
3. there is no subset of that has properties 1 and 2 and contains $E^1(\lambda^*, \tau) \times E^2(\lambda^*, \tau)$ as a proper subset.

The set

$$\left\{E^1(\lambda^*, \tau) \times E^2(\lambda^*, \tau) : \max\left\{\tau^1_{\min}, \tau^2_{\min}\right\} < \tau < \min\left\{\tau^1_{\max}, \tau^2_{\max}\right\}\right\}$$

is a partition of $F_{NF}^{-1}(\lambda^*)$ (I will leave you to show this). Notice that the covering is indexed by the messages $(m_1, m_2) = (\lambda, \tau)$.

Now we take the labeling of the rectangles to construct the equilibrium message correspondence. Let

$$m_1 = \lambda,\ m_2 = \tau.$$

And let

$$\begin{aligned} \mu_S^1(a_1, a_2) &= \{(m_1, m_2) \in M : P_1(m_1, a_1, a_2) = m_2\}, \\ \mu_S^2(b_1, b_2) &= \{(m_1, m_2) \in M : P_2(m_1, b_1, b_2) = m_2\}. \end{aligned} \tag{23}$$

Define

$$\mu_S(a_1, a_2, b_1, b_2) = \mu_S^1(a_1, a_2) \cap \mu_S^2(b_1, b_2);$$

notice that

$$\mu_S(a_1, a_2, b_1, b_2) = \{(m_1, m_2) \in M : P_1(m_1, a_1, a_2) = P_2(m_1, b_1, b_2)\}.$$

For $(m_1, m_2) \in \mu_S(a_1, a_2, b_1, b_2)$, define $h_S(m_1, m_2) = m_1 = \lambda^*$. Notice that the equilibrium message space has dimension two. The mechanism $\pi_S = (\mu_S, M_S, h_S)$ realizes the goal function F_{NF} defined at (21).

REFERENCES

Hurwicz, On informationally decentralized systems, in McGuire and Radner (eds), *Decision and Organization*, 1972.

Hurwicz, Incentive aspects of decentralization, in *Handbook of Mathematical Economics,* v. 3.

Hurwicz and Reiter.