ECN 612 - HANDOUT - FALL 2009
Public Goods

THE ALGEBRA OF PARETO OPTIMALITY

We assume two consumers and two produced goods, one of which is public and the other private. Let

$$
\begin{aligned}
x &= \text{public good} \\
y_i &= \text{individual } i\text{'s consumption of the private good, } i = A, B \\
u^i &= \text{utility function of individual } i,\ i = A, B \\
L_j &= \text{amount of the input used in the production of } j,\ j = x, y \\
\bar{L} &= \text{given amount of the input.}
\end{aligned}
$$

To find a Pareto optimum point, we solve the following problem.

$$
\begin{array}{rl}
\max\limits_{x, y_A, y_B, L_x, L_y} & u^A(x, y_A) \\
\text{st} & u^B(x, y_B) = \bar{u}^B \\
& x = f(L_x) \\
& y_A + y_B = g(L_y) \\
& L_x + L_y = \bar{L}
\end{array}
$$

Let

$$\lambda \equiv (\lambda_1, \lambda_2, \lambda_3, \lambda_4); MU_x^i = u_x^i, MU_y^i = u_{y_i}^i, i = A, B; MP^x = f'(L_x), MP^y = g'(L_y).$$

The Lagrangian is

$$
\begin{aligned}
\mathcal{L}(x, y_A, y_B, L_x, L_y, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \quad = \quad & u^A(x, y_A) + \lambda_1\left[u^B(x, y_B) - \bar{u}^B\right] + \lambda_2\left[f(L_x) - x\right] + \\
& \lambda_3\left[g(L_y) - y_A - y_B\right] + \lambda_4\left(\bar{L} - L_x - L_y\right).
\end{aligned}
$$

We assume the following conditions are evaluated at the solution

$$\left(x^*, y_A^*, y_B^*, L_x^*, L_y^*, \lambda_1^*, \lambda_2^*, \lambda_3^*, \lambda_4^*\right).$$

PARTIAL SET OF FOC

$$\mathcal{L}_x = MU_x^A + \lambda_1 MU_x^B - \lambda_2 = 0 \tag{1}$$

$$\mathcal{L}_{y_A} = MU_y^A - \lambda_3 = 0 \tag{2}$$

$$\mathcal{L}_{y_B} = \lambda_1 MU_y^B - \lambda_3 = 0 \tag{3}$$

$$\mathcal{L}_{L_x} = \lambda_2 MP^x - \lambda_4 = 0 \tag{4}$$

$$\mathcal{L}_{L_y} = \lambda_3 MP^y - \lambda_4 = 0 \tag{5}$$

Use (4) and (5) to find that

$$\frac{\lambda_2 MP^x}{\lambda_3 MP^y} = \frac{\lambda_4}{\lambda_4} \Rightarrow \frac{\lambda_2}{\lambda_3} = \frac{MP^y}{MP^x} = MRT_{xy}. \tag{6}$$

Use (2) and (3) to find that

$$\frac{MU_y^A}{\lambda_1 MU_y^B} = \frac{\lambda_3}{\lambda_3} \Rightarrow \frac{MU_y^A}{MU_y^B} = \lambda_1. \tag{7}$$

Substitute (7) into (1) to find that

$$MU_x^A + \frac{MU_y^A}{MU_y^B} \cdot MU_x^B = \lambda_2 \Rightarrow \lambda_2 = MU_x^A + MU_y^A \cdot \frac{MU_x^B}{MU_y^B}. \tag{8}$$

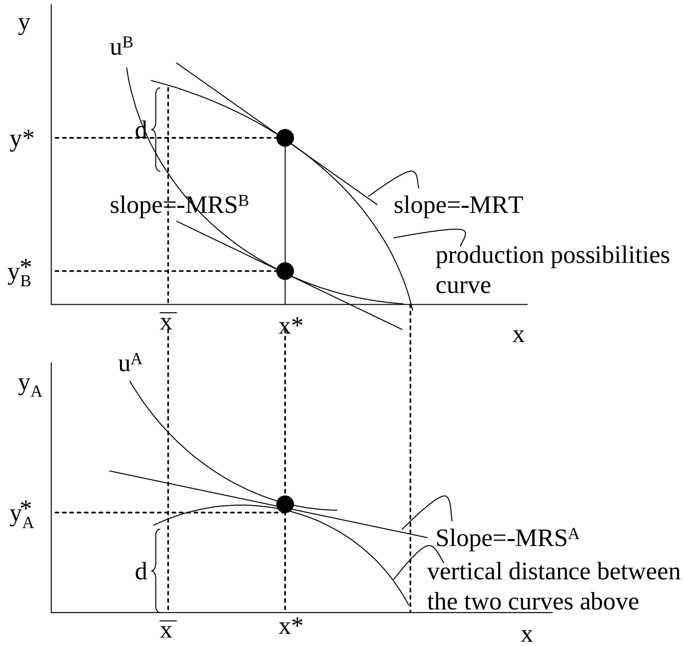
Use (2) and (8) to find that

$$\frac{\lambda_2}{\lambda_3} = \frac{MU_x^A + MU_y^A \cdot \dfrac{MU_x^B}{MU_y^B}}{MU_y^A} = \frac{MU_x^A}{MU_y^A} + \frac{MU_x^B}{MU_y^B} = MRS_{xy}^A + MRS_{xy}^B. \tag{9}$$

Now use (6) and (9) to find that at the optimum,

$$MRT_{xy} = MRS_{xy}^A + MRS_{xy}^B.$$

I'll leave you to check SOC.



PROBLEM FOR YOU

Assume that $MRT_{xy} = 1$. Assume that consumers have the utility functions

$$u^i(x, y_i) = \beta_i \log x + y_i, \tag{10}$$

$i = 1, \ldots, I$. Show that the ith consumer's marginal rate of substitution is

$$\frac{\beta_i}{x}$$

and the optimal level of public good is

$$x^* = \sum_{j=1}^{I} \beta_j. \tag{11}$$

THE LINDAHL MECHANISM

Assume here and in the rest of the handout that $p_y = 1$. Under this mechanism, consumers pay personalized prices, called Lindahl prices,

$$p_i^* = MRS_{xy}^i \left(x^*, y_i^*\right).$$

Note that in the problem of the last section, the Lindahl prices are

$$p_i^* = \frac{\beta_i}{\sum_{j=1}^{I} \beta_j}. \tag{12}$$

UNDERPROVISION OF THE PUBLIC GOOD UNDER THE LINDAHL MECHANISM

We assume that consumers have utility functions like those in the problem of the next-to-the-last section. We assume that individual i believes that everyone else will truthfully reveal their preferences (which he/she knows). We determine what declaration of β_i, denoted by B_i, maximizes i's utility. Individual i's budget line is

$$p_i x + y_i = w_i$$

or

$$y_i = w_i - p_i x. \tag{13}$$

From (11), we find that

$$x = B_i + \sum_{j \neq i} \beta_j. \tag{14}$$

From (12), we find that

$$p_i = \frac{B_i}{B_i + \sum_{j \neq i} \beta_j}. \tag{15}$$

Substitute (14) and (15) into (13) to find that individual i's budget line is

$$y_i = w_i - B_i. \tag{16}$$

Substitute (14) and (16) into (10) to find that

$$f(B_i) = \beta_i \log \left(B_i + \sum_{j \neq i} \beta_j \right) + w_i - B_i$$

is the function individual i will maximize WRT B_i. You should find that

$$B_i^* = \beta_i - \sum_{j \neq i} \beta_j < \beta_i;$$

this shows that individual i has the incentive to underdeclare his/her MRS_{xy}^i when he/she assumes all others will tell the truth. It follows that the public good will be underprovided. And we have shown that the Lindahl Mechanism is not incentive compatible.

THE CLARKE TAX MECHANISM

Assume that

$$u^i(x, y_i) = v_i(x) + y_i, \tag{17}$$

$i = 1, \ldots, I$. Individuals submit $\tilde{v}_i(x)$ (their declared valuations, possibly false). Given these, the government solves the problem

$$\max_x \sum_{j=1}^{I} \tilde{v}_j(x) - x, \tag{18}$$

which means that the government finds the x such that

$$\sum_{j=1}^{I} \tilde{v}_j'(x) = 1.$$

Individual i, $i = 1, \ldots, I$, is assessed the tax

$$T_i = x - \sum_{j \neq i} \tilde{v}_j(x). \tag{19}$$

The individual's budget constraint is

$$y_i + T_i = w_i$$

or

$$y_i = w_i - T_i. \tag{20}$$

Substitute (19) into (20) and the result into (17) to find that individual i would like the following to be maximized:

$$\hat{U}^i = v_i(x) + w_i - \left[x - \sum_{j \neq i} \tilde{v}_j(x) \right]$$

or

$$U^i = v_i(x) - \left[x - \sum_{j \neq i} \tilde{v}_j(x) \right]. \tag{21}$$

Recall from (18) that the government solves the problem

$$\max_x \tilde{v}_i(x) - \left[x - \sum_{j \neq i} \tilde{v}_j(x) \right]. \tag{22}$$

Therefore, comparing (21) and (22), we see that if the individual submits his/her true valuation, then the government maximizes the function that the individual would like to see maximized. That is, we have argued that no matter whether others lie or tell the truth, individual i has the incentive to tell the truth; more formally, truth-telling is a dominant strategy.

THE GROVES MECHANISM

This mechanism is very much like the Clarke Mechanism, except an effort is made to correct for the problem of deficits We look at a particular Groves scheme. Consider the possibility of adding the following term to T_i.

$$S_i = \max_x \sum_{j \neq i} \left(\tilde{v}_j(x) - \frac{x}{I} \right).$$

Note that

$$\sum_{j \neq i} \left(\tilde{v}_j(x) - \frac{x}{I} \right)$$

can be interpreted as the aggregate net benefits to all individuals except i when each individual pays an equal amount for the public good. Notice that S_i does not depend on $\tilde{v}_i$ or on the x chosen by the government. Now let

$$T_i = \underbrace{x - \sum_{j \neq i} \tilde{v}_j(x)}_{\text{Clarke Tax}} + S_i.$$

Because of the properties of S_i, truth should still be the dominant strategy. Furthermore,

$$\begin{aligned} T_i &= x - \sum_{j \neq i} \tilde{v}_j(x) + S_i \\ &= x - \sum_{j \neq i} \tilde{v}_j(x) + \max_x \sum_{j \neq i} \left(\tilde{v}_j(x) - \frac{x}{I} \right) \\ &= \frac{x}{I} - \sum_{j \neq i} \left(\tilde{v}_j(x) - \frac{x}{I} \right) + \max_x \sum_{j \neq i} \left(\tilde{v}_j(x) - \frac{x}{I} \right) \\ &\geq \frac{x}{I}. \end{aligned}$$

It follows that

$$\sum_{i=1}^{I} T_i \geq x,$$

i.e., the taxes will finance the public good.

THE GROVES-LEDYARD MECHANISM

We present a specific Groves-Ledyard scheme. We continue to assume that utility is quasilinear (they don't), and we present a more specific tax than they present. Here individuals send to the government desired additions (or reductions) to the amounts of the public good requested by others (denoted a_i).

The government sets the level of public good by letting

$$x = \sum_{i=1}^{I} a_i.$$

Taxes are determined by the following.

$$T = \frac{x}{I} + \tfrac{1}{2}\gamma \left[\frac{I-1}{I} \left(a_i - \mu_{)i(} \right)^2 - \sum_{j \neq i} \frac{1}{I-2} \left(a_j - \mu_{)i(} \right)^2 \right],$$

where

$$\mu_{)i(} = \frac{1}{I-1} \sum_{j \neq i} a_j$$

and γ is a positive number. Now assume that individual i takes $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_I$ as given. He/she will choose a_i to maximize

$$v_i \left(a_i + \sum_{j \neq i} a_j \right) - T_i$$

or

$$v_i \left(a_i + \sum_{j \neq i} a_j \right) - \frac{1}{I} \left(a_i + \sum_{j \neq i} a_j \right) - \tfrac{1}{2}\gamma \left[\frac{I-1}{I} \left(a_i - \mu_{)i(} \right)^2 - \sum_{j \neq i} \frac{1}{I-2} \left(a_j - \mu_{)i(} \right)^2 \right].$$

The FOC for his/her problem is

$$v_i' \left(a_i^* + \sum_{j \neq i} a_j \right) - \frac{1}{I} - \gamma \cdot \frac{I-1}{I} \left(a_i^* - \mu_{)i(} \right) = 0 \Rightarrow$$

$$v_i' \left(a_i^* + \sum_{j \neq i} a_j \right) = \frac{1}{I} + \gamma \cdot \frac{I-1}{I} \left(a_i^* - \mu_{)i(} \right).$$

A *Groves-Ledyard equilibrium* is a list of additions $(a_1^*, \ldots, a_I^*)$ and an output level

$$x^* = \sum_{i=1}^{I} a_i^*$$

such that a_i^* maximizes individual i's utility given a_j^*, $j \neq i$, $j = 1, \ldots, I$.

COURNOT-LINDAHL MECHANISM

Notation

Goods: $x =$ private good, $y =$ public good

Consumers: For $i = 1, \ldots, I$,

(i) $\omega_i > 0$: endowment of private good

(ii) $X_i \subseteq \mathbf{R}_+^2$ individual i's consumption set, $(\omega_i, 0) \in X_i$

(iii) $u_i : X_i \to \mathbf{R}$

Production: $y = \frac{1}{\beta} \cdot x$, where $\beta > 0$

Feasibility constraint:

$$\sum_{i=1}^{I} x_i + \beta y \leq \sum_{i=1}^{I} \omega_i$$

Definition 1 *An allocation is an $(I+1)-$tuple $(y, x_1, \ldots, x_I) \in \mathbf{R}_+ \times \mathbf{R}_+^2$.*

Definition 2 *A government is defined by specifying*

(a) for each player $i \in \mathcal{I} \equiv \{1, \ldots, I\}$, the set M of messages that player i is allowed to report.

(b) the $I+1$ outcome functions $\mathcal{Y} : \Pi_1^I M_i \to \mathbf{R}_+$ and $\tau_i : \Pi_1^I M_i \to \mathbf{R}$, $i \in \mathcal{I}$, which specify, respectively, the public level y and the taxes t_i, $i \in \mathcal{I}$, all as functions of the $I-$tuple $\mathbf{m} = (m_1, \ldots, m_I)$ of messages.

Definition 3 *The Cournot-Lindahl government is defined by*

(a) $M_i = \mathbf{R}$, $i \in \mathcal{I}$

(b) $\mathcal{Y}(\mathbf{m}) = \sum_{1}^{I} m_i$

(c) $\tau_i(\mathbf{m}) = \left(\frac{1}{I} \cdot \beta + m_{i+2} - m_{i+1}\right) \mathcal{Y}(\mathbf{m})$, *$i \in \mathcal{I}$, where the subscripts $I+1$ and $I+2$ are understood as "modulo I," i.e., $I+1 = 1$, $I+2 = 2$.*

Definition 4 *A Cournot-Lindahl equilibrium is a list $\left((x_i^*, m_i^*)_{i=1}^I, y^*\right)$ satisfying the conditions*

(a) $\sum_{i=1}^{I} x_i^* + \beta y^* = \sum_{i=1}^{I} \omega_i$

(b) $\sum_{i=1}^{I} m_i^* = y^*$

(c) $(x_i^*, y^*) \in X_i$, $i \in \mathcal{I}$

(d) (x_i^*, y^*) maximizes u_i among all $(x_i, y) \in X_i$ for which there is some $m_i \in \mathbf{R}$ satisfying

$$m_i + \sum_{j \neq i} m_j^* = y$$

and

$$x_i + \tau_i \left(m_i, m_{)i(}^*\right) \leq \omega_i,$$

for every $i \in \mathcal{I}$.

Definition 5 *A Lindahl equilibrium is a list $\left((q_i^*)_{i=1}^I, (x_i^*)_{i=1}^I, y^*\right)$ satisfying the following conditions.*

(a) $\sum_{i=1}^{I} x_i^* + \beta y^* = \sum_{i=1}^{I} \omega_i$

(b) $(x_i^*, y^*) \in X_i$, $i \in \mathcal{I}$

(c) For every $i \in \mathcal{I}$, (x_i^*, y^*) maximizes u_i among all $(x_i, y) \in X_i$ that satisfy

$$x_i + q_i^* y \leq \omega_i.$$

Theorem 6 *In an economy with three or more consumers, there is a one-to-one correspondence between the set of Cornot-Lindahl equilibria and the set of Lindahl equilibria. In the equilibria which correspond to one another the allocations are identical and the Lindahl prices q_i and the individuals' Cournot-Lindahl messages m_i are related via*

$$q_i = \frac{1}{I} \cdot \beta + m_{i+2} - m_{i+1}.$$

Proof. First, we show that a Cournot-Lindahl equilibrium allocation is a Lindahl equilibrium allocation. Suppose that each player behaves according to the Cournot-Nash assumption, treating the other $I - 1$ players' messages as given while he/she chooses his/her own message. The player i can by his/her choice of m_i demand any public good level that he/she can afford, but for each unit of public good he/she must pay a price

$$q_i = \frac{1}{I} \cdot \beta + m_{i+2} - m_{i+1}.$$

Furthermore, since the player is taking others' messages as given, he/she is taking his own price as given. Since $\sum_{i=1}^{I} q_i = \beta$, any equilibrium $I-$ tuple $\mathbf{m} = (m_1, \ldots, m_I)$ will correspond to a Lindahl equilibrium. The resulting allocation will be a Lindahl allocation and the resulting prices $(q_1, \ldots, q_I)$ will be a vector of Lindahl prices.

Now we argue that a Lindahl equilibrium allocation is a Cournot-Lindahl allocation. Suppose that $(q_1, \ldots, q_I)$ is a Lindahl price vector and that $(y, x_1, \ldots, x_I)$ is the corresponding Lindahl allocation. Consider the following system of I linear equations in the I variables $m_1, \ldots, m_I$.

$$\begin{aligned} m_1 + m_2 + \cdots + m_I &= y \\ m_2 - m_1 &= q_I - \frac{1}{I} \cdot \beta \\ m_3 - m_2 &= q_1 - \frac{1}{I} \cdot \beta \\ &\vdots \\ m_I - m_{I-1} &= q_{I-2} - \frac{1}{I} \cdot \beta \end{aligned}$$

The latter system clearly has a solution $(m_1^*, \ldots, m_I^*)$ and it is clearly a Cournot-Nash equilibrium. ■

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