Variety: Consumer Choice and Optimal Diversity

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Abstract

Consumers choose from among the varieties of two brands and an outside good using order statistics. We analytically derive demand functions conditional on their valuations of the varieties being distributed independently uniform. Based on this theory, we estimate a three-parameter empirical version of the model for the soft-drink market. These estimates are used to determine the effects of changes in the number of varieties on demand curves and consumer welfare. We use our estimates to compare the profit-maximizing number of varieties within a grocery store to the socially optimal number and find that consumer surplus and welfare would increase with more variety.

key words: varieties, product line length, consumer surplus, welfare, demand, order statistics, oligopolistic

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Variety: Consumer Choice and Welfare

1. Introduction

How do brands' product lengths—the number of varieties that each sells—affect consumers' brand choice? Are there too few or too many varieties? To address these questions, we use order statistics to develop a new theory of consumer choice across brands that have many varieties. After picking the best variety within each brand (an order statistic), the consumer then selects the best choice across the brands (an order statistic over order statistics) and compares that best choice to an outside good. Given that consumers' tastes are distributed uniformly, we derive a complete set of analytic results. To illustrate the theory, we apply this model to the soft-drink market. A terse, three parameter empirical version of the model does very well in describing actual consumer choice. We use these estimates to determine the effects of changes in the number of varieties on demand curves and consumer welfare and to address other welfare questions.

According to many food and beverage manufacturing executives, brands are maintained through product differentiation (e.g., Nijssen and Van Trijp, 1998). Firms constantly innovate to keep up with changing consumer tastes.[1] Products that are not accepted by consumers are quickly dropped. One might think of this approach of constantly providing new products as a flagpole strategy: "Let's run it up the flagpole and see who salutes it." Firms differentiate by changing flavors or other aspects of the product as well as by altering the size or shape of the container.

We examine a market in which each of two firms (brands) produce many varieties of a good. Examples of such markets include sporting goods, yogurt, ice cream, and beverages. Sporting good firms that produce a variety of balls, gloves, shoes that differ only slightly in terms of which athlete endorses them or a variety of aesthetic bells and whistles. Yogurts vary by flavor, whether the fruit is on the bottom, and in other ways. Ice creams vary by flavor and fat content. Beverage manufacturers offer many varieties:

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[1] For example in 2000, Snapple introduced a new fruit drink, Diet Orange Carrot Fruit Drink ("Fruit Beverages Scope," *Beverage World*, February, 2000, p. 26), presumably reasoning that if they can sell that flavor, they can sell any flavor.

For example, Coca-Cola HBC, one of the largest bottlers of non-alcoholic beverages in Europe, sells 147 carbonated and 461 non-carbonated beverages across the water, juice and juice drinks, energy drinks, sports drinks, and ready-to-drink iced tea and coffee categories.

Our empirical example concerns the competition between Coca-Cola and Pepsi in the U.S. soft-drink market, which are the dominant oligopolistic firms. According to *Beverage Digest*, the two companies accounted for three-quarters of the U.S. carbonated beverage market in 1999, the sample period for our empirical analysis. Coca-Cola sells many variations of its flagship product, including Coca-Cola, Cherry Coke, Diet Coke, Caffeine-Free Coke, Caffeine-Free Diet Coke, and Black Cherry Vanilla Coke. It also sells or has sold a wide variety of other soft drinks, including Tab, Sprite, Fresca, Fanta, Barq's Root Beer, Mello Yello, and Pibb Xtra.[2] Retailers differ as to how many of these varieties they carry, but no retailer carries all of them. (Note: For simplicity in the following, we refer to the Coke brand and the Pepsi brand, whereas each company actually has several brands with many varieties in each.)

Most existing theoretical work on product differentiation or product length focuses on firms' behavior rather than on consumers' choices. Much of the theoretical literature abstracts from how consumers choose and assumes that there are only a small number of varieties (e.g., Brander and Eaton 1984, Gilbert and Matutes 1993, and Villas-Boas 2004).

Other theoretical papers model demand using specific functional forms that depend on the total number of varieties. By employing explicit utility functions, they can analyze the welfare effects of greater variety.

Four classic papers on product differentiation—Dixit and Stiglitz (1977), Spence (1976), Salop (1979), and Deneckere and Rothschild (1992)—assumed that each monopolistically competitive firm produces a single product and then asked if there are too many or too few products. The Chamberlin-representative-consumer competition

[2] Coke's website claims the company sells 700 low-calorie or no-calorie drinks throughout the world. Some of Coca-Cola's other beverages include Vault and Sugar Free Full Throttle energy drinks, Enviga, Gold Peak tea, Fruitopia fruit drinks, Powerade sports drinks, Dasani and Vitaminwater, Zu and Caribou coffees, Nestea products, Bacardi mixers, and various Minute Maid beverages.

papers of Dixit and Stiglitz (1977) and Spence (1976) used a constant elasticity of substitution (CES) utility function for a representative consumer that depends explicitly on the number of products. In the Hotelling-competition model of Salop (1979), consumers' tastes are uniformly distributed around a circle and products are evenly spaced around the circle (adjusting as new firms enter). Deneckere and Rothschild (1986) nested what they called a Chamberlin model (Perloff and Salop 1985) and the Hotelling-circle model of Salop (1979).

Some more recent papers modified these models to look at firms with product lines rather than firms that produce only one good each, but they maintain these functional assumptions. For example, Raubitschek (1987) extends the Spence-CES model to allow for brands to have varieties, and Klemperer (1992) modified the circle model to allow for an endogenous determination of spacing of varieties.

Kim et al. (2002) chose a CES-like functional form for utility with a log-normal random utility component to estimate the compensating variation from removing an existing brand of yogurt. Draganska and Jain (2005) employed the Kim et al. model and assumed that all flavors yield the same utility, all flavors are offered with equal probability, and the cost of evaluating each flavor is convex.

The rest of the empirical literature falls loosely into three categories: flexible demand systems, nested or mixed logit, and demand systems that add measures of the length of product lines. Hausman (1996) used a flexible system of demand equations to estimate the welfare effects of adding one brand of cereal by assuming that the price went from infinity to a finite level. Kadiyali et al. (1999) employed a linear system of demands curves to study product extensions effects on prices. Israilevich (2004) estimated an AIDS system and concluded that a grocery store chain carries the optimal number or too many products. Nevo (2000) employed a mixed logit approach to address a similar question to Hausman's about the effects of adding a product. Bayus and Putsis (1999) estimated a three-equation system for share, price, and product-line length to examine the effect of product-line length on price and market share.

Our work is closest to that of Perloff and Salop (1985) and Anderson et al. (1992), who modeled the effects of greater product diversity on prices, market share, and welfare. (These models have been used as the basis for mixed logit empirical studies.) In those models and in the current model, consumers place a value on each product and then choose the variety with the largest net surplus: the consumer’s value minus the price. The values are drawn independently from a distribution. Each consumer buys one unit of one of these varieties if the net surplus from the consumer’s favorite exceeds that of an outside good.

There are two main differences between our model and those of Perloff-Salop and Anderson et al. First, these earlier models assumed that each manufacturing firm produced a single product, whereas our model has two manufacturing firms (brands) each of which produces multiple varieties. Second, rather than focus on the decision of manufacturers, we examine the decision of the retailer. The reason for this later difference is that, at least in grocery markets, retailers determine the number of varieties to carry rather than manufacturers, who produce a much larger number of varieties to the market as a whole.

We start by providing intuition for our model by examining a monopoly firm and by providing a simplified example of a duopoly problem. Next, we use order statistics to derive a model of how consumer choice varies with brands’ product lengths and derive a number of analytic, comparative statics properties. We use our model to estimate a demand system for Coke, Pepsi, and an outside good at supermarkets. Using our estimated demand system, we simulate various demand and consumer surplus comparative statics results. We then use a simulation to determine whether grocery stores provide the optimal number of varieties. We compare the profit-maximizing equilibrium to the social optima where the social planner can set both prices and the number of varieties, only prices, or only the number of varieties.

2. Varieties and Consumer Choice

In our model, each consumer buys one unit of a good by choosing among all the available varieties offered by both brands. The value that a consumer places on each variety is drawn independently from uniform distributions that may differ across brands. Each consumer picks the variety with the largest net surplus if that net surplus is greater than the net surplus provided by the outside good.

We concentrate on soft drinks sold by Coke and Pepsi, where each variety of a brand sells for the same price, though the price may differ across brands. Before plunging into our order-statistics model with two brands and an outside good, we illustrate the basic idea with two simpler examples.

One Brand

Initially, suppose that there is a monopoly brand that produces a variety of flavors, consumers choose their favorite variety, and the net surplus from the outside good is zero. A typical consumer places a value on each of the n varieties that is drawn independently from a uniform [0, 1] distribution. The price, p, lies within the range (0, 1) by appropriate scaling. The probability is $1 - p$ that a consumer will place a higher value on a given variety than its price. The probability that at least one variety is more valuable to the consumer than its price is 1 minus the probability that no variety has a value greater than the price: $1 - (1 - [1 - p])^n = 1 - p^n$.

The aggregate demand curve is the number of consumers, Z, multiplied by this probability: $[1 - p^n]Z$. (For simplicity, we henceforth normalize Z to equal one.) The slope of the demand curve with respect to price is $-np^{n-1} < 0$. If the number of varieties increases from n to $n + 1$, then the quantity purchased increases by $(1 - p)p^n$, which is positive because $p \in (0, 1)$. As n gets large, virtually everyone buys a variety from this brand.

Two Brands

Now, suppose that there are two brands. The consumer might use a three-step procedure to pick which variety if any to buy. The consumer first picks the highest net-surplus variety within each brand, then the consumer selects between the two top choices for each brand, finally the consumer compares the best overall variety choice across the brands with the net surplus the consumer places on an outsider product. If the best variety across brands is more attractive than the outside good, the consumer buys that variety.

We can use a table to illustrate the effects from adding one more variety for one brand. The numbers in the table are the net surplus a consumer obtains from each variety. The net surplus from the outside good is .5. Initially, the store carries two varieties of Coke and two variety of Pepsi (the first two columns of the Pepsi section of the table).

The first consumer (row one) receives a net surplus of .9 from the first variety of Coke and .6 from the second variety, so the consumer prefers the first variety. Similarly that consumer prefers the first Pepsi variety (.3) to the second one (.1). This consumer will buy one unit of Coke because the net surplus from the preferred variety of Coke (.9) exceeds the net surplus from the best Pepsi option (.3) and the net surplus from the outside good (.5). In the table, the consumer's overall choice is indicated by expressing the relevant surplus in italics (ignore the bolding for the moment).

	Outside	**Coke**		**Pepsi**		
Consumer		*First*	*Second*	*First*	*Second*	*Third*
1	.5	***.9***	.6	.3	.1	.4
2	.5	.7	.6	.6	***.8***	.7
3	.5	.4	*.8*	.3	.6	**.9**
4	.5	.6	.3	***.7***	.4	.1
5	*.5*	.1	.2	.3	.2	**.6**

By similar reasoning, Consumer 2 buys the second variety of Pepsi, Consumer 3 buys the second variety of Coke, Consumer 4 chooses the first variety of Pepsi, and Consumer 5 opts for the outside good. In this market, 40% (2 out of 5) choose Coke, 40% choose Pepsi, and 20% consumes the outside good.

Now suppose that the store starts carrying a third variety of Pepsi. In the table, the net surplus corresponding to the consumer's choice is in bold type. The extra variety affects the decisions of only Consumers 3 and 5. Consumer 3 switches from buying the second variety of Coke to the third variety of Pepsi. Consumer 5 changes from the outside good to the third variety of Pepsi.

Pepsi's market share rises with the addition of another variety. The market shares are now 80% Pepsi and 20% Coke (though we wouldn't expect such an extreme outcome in general).

Consumers 1, 2, and 4 are unaffected by the additional variety, while Consumers 3 and 5 are better off, so total consumer surplus must rise. Initially, the total surplus was .9 + .8 + .8 +. 7 + .5 = 3.7. After the third variety of Pepsi is introduced, total surplus is .9 + .8 + .9 +. 7 + .6 = 3.9. This example illustrates that consumers benefit—have higher consumer surplus—from more choice where we hold price constant.

3. Order-Statistics Model

We now turn to a formal analysis of our model. We develop the model in four steps. First we discuss how a consumer would compare two sets of varieties if prices were zero and there is no outside good. Second, we introduce non-zero prices. Third, we allow the value distribution for each brand to have a different support, so that consumers might prefer one brand to another on average. Fourth, we introduce an outside good with a non-negative net surplus.

Distribution of the Difference of Independent Maxima

Each consumer's valuation of any variety of Brand 1 (Coca-Cola) or Brand 2 (Pepsi) is drawn independently from uniform distributions on [0, θ] with independent random sample of sizes n_1 and n_2 respectively, where n_1 is the number of varieties offered by Brand 1 (Coke, Diet Coke, …) and n_2 is the number of varieties offered by Brand 2 (Pepsi, Diet Pepsi,…). Let L_1 and L_2 be the maximal observations of a consumer's valuation of varieties for Brand 1 and Brand 2. That is, $L_1 = \max(L_{1,1},...,L_{1,n_1})$ and $L_2 = \max(L_{2,1},...,L_{2,n_2})$, where the valuations $L_{1,j}$ and $L_{2,j}$ are distributed independently uniform on $[0,\ \theta]$.

The distribution of the maximal valuation difference, $L_1 - L_2$, is the probability that a consumer selects Coke or Pepsi, or the market shares (relative demand): $s_1 \in [0,\ 1]$ and $s_2 \in [0,\ 1]$. For now, everyone buys one unit of either Coke or Pepsi—there is no outside good—so that $s_1 + s_2 = 1$. Also for now, we ignore prices. For example, suppose that a firm provides a free soft-drink at lunch, so that its employees simply have to decide which variety of which brand to choose independent of price. Using standard notation where ${}_aC_b = a!/(a-b)!/b!$, we derive

Proposition 1. The probabilities that the consumer chooses Brand 1 and Brand 2 are (respectively):

$$s_1(n_1,n_2) = \Pr(L_1 > L_2, n, m) = \sum_{j=1}^{n_1} (-1)^{j-1} \frac{{}_{n_1}C_j}{{}_{n_2+j}C_j},$$

$$s_2(n_1,n_2) = \Pr(L_1 \le L_2, n, m) = \sum_{j=1}^{n_2} (-1)^{j-1} \frac{{}_{n_2}C_j}{{}_{n_1+j}C_j}. \quad \blacksquare$$

The proof is in the Appendix. The share functions are finite hypergeometric sums, which can be reformulated in terms of the hypergeometric function (Gauss 1813). In economic terms, cumulations in Proposition 1 are the probabilities of choosing Coke ($L_1 > L_2$) or Pepsi ($L_1 < L_2$), where the consumer's objective is to pick the Coke or Pepsi variety with the highest value (given that there is no outside good or price).

Using the Chu-Vandermonde identity for hypergeometric functions (Andrews et al., 2001, p.67), the following is true:

Proposition 2. The shares in Proposition 1 can be reformulated as:

$$s_1(n_1, n_2) = \frac{n_1}{n_1 + n_2},$$
$$s_2(n_1, n_2) = \frac{n_2}{n_1 + n_2}.$$

■

For notational simplicity, we will often drop the arguments of the share functions in the following. Murty (1955) derived a similar result when he considered the distribution of L_1 / L_2 and calculated the probabilities:

$$\Pr\left(\frac{L_1}{L_2} > 1\right) = \frac{n_1}{n_1 + n_2},$$
$$\Pr\left(\frac{L_1}{L_2} \le 1\right) = \frac{n_2}{n_1 + n_2}.$$

In the next section, we want to examine net surpluses, where we subtract prices from L_1 and L_2 and recalculate cumulations in Proposition 1. Doing so is much easier if we compare the differences between L_1 and L_2 rather than their ratio. Relocation of a random variate by a constant is generally simpler than rescaling.

Relocation of the Distribution of the Difference by Prices

We now introduce prices, which we assume are exogenously determined, which would be true if grocery stores are price takers with respect to soft drinks. Let $p_1 > 0$ be the price for all varieties of Coke and $p_2 > 0$ be the price for all varieties of Pepsi. It is well-documented (e.g., Draganska and Jain, 2005) that retailers set the same retail prices for all varieties of a specific brand of a number of grocery products such as yogurt or soft drinks. The maximal net surplus of the choices are $\ell_1 = L_1 - p_1$ for Coke and $\ell_2 = L_2 - p_2$ for Pepsi. Defining the price difference as $\pi = (p_1 - p_2)$, we have

Proposition 3. Cumulations (net of prices) above and below zero are:

$$\tilde{s}_1(n_1, n_2, p_1, p_2) = \sum_{j=1}^{n_1} (-1)^{j-1} \frac{{}_{n_1}C_j}{{}_{n_2+j}C_j} \left(1 - \frac{\pi}{\theta}\right)^{n_2+j} \text{ for } p_1 > p,$$

$$\tilde{s}_2(n_1, n_2, p_1, p_2) = \sum_{j=1}^{n_2} (-1)^{j-1} \frac{{}_{n_2}C_j}{{}_{n_1+j}C_j} \left(1 + \frac{\pi}{\theta}\right)^{n_1+j} \text{ for } p_1 < p_2.$$

■

The proof is in the Appendix. If $p_1 > p_2$, then $\tilde{s}_1$ is the share of Coke purchased, and $1 - \tilde{s}_1$ is the share of Pepsi. If $p_1 < p_2$, then $1 - \tilde{s}_2$ is the share of Coke, and $\tilde{s}_2$ is the share of Pepsi. Obviously, when $p_1 = p_2$, the shares in Proposition 3 are identical to those in Proposition 1. If preferences are identical across consumers and the population size of consumers is known, then multiplying the shares in Proposition 3 times the population produces demand curves for Coke and Pepsi, respectively. Therefore, these probabilities are relative demands or market shares.

If we constrain π/θ to the unit circle—which is equivalent to imposing the price constraint that $p_1, p_2 \in (0, \theta)$—then $(1 - \pi/\theta)^{n_2+j}$ and $(1 + \pi/\theta)^{n_1+j}$ are on the unit interval and the probabilities in Proposition 3 are weighted versions of the shares in Proposition 1 such that, for $-1 < \pi/\theta < 1$,

$$\tilde{s}_1 < s_1 \text{ and } \tilde{s}_2 > s_2 \text{ for } p_1 > p_2,$$

$$\tilde{s}_2 < s_2 \text{ and } \tilde{s}_1 > s_1 \text{ for } p_1 < p_2.$$

That is, under this constraint, if we set $p_1 = p_2$ in $\tilde{s}_1$ and $\tilde{s}_2$, we have $\tilde{s}_1 = s_1 = \tilde{s}_2 = s_2$. If Coke's price rises above Pepsi's price, then the share of Coke falls ($\tilde{s}_1 < s_1$) and the share of Pepsi rises $\tilde{s}_2 > s_2$.

The derivatives of the relative shares with respect to the continuous variables are straight-forward and the results are intuitively appealing. For $p_1 > p_2$,

$$\frac{\partial \tilde{s}_1}{\partial p_1} = -\frac{1}{\theta}\sum_{j=1}^{n_1}(-1)^{j-1}\frac{{}_{n_1}C_j}{{}_{n_1+j}C_j}\frac{1}{n_2+j}\left(1-\frac{\pi}{\theta}\right)^{n_2+j-1} \le 0,$$
$$\frac{\partial \tilde{s}_1}{\partial p_1} = -\frac{1}{\theta}\frac{\partial \tilde{s}_1}{\partial p_2} = \frac{\partial \tilde{s}_2}{\partial p_1} \ge 0,$$
$$\frac{\partial \tilde{s}_1}{\partial \theta} = \frac{\pi}{\theta^2}\sum_{j=1}^{n_1}(-1)^{j-1}\frac{{}_{n_1}C_j}{{}_{n_1+j}C_j}\frac{1}{n_2+j}\left(1-\frac{\pi}{\theta}\right)^{n_2+j-1} \ge 0.$$

Similarly for $\tilde{s}_2$. The market share for Coke is decreasing in the price of Coke, increasing in the price of Pepsi, and increasing in the maximal gross value, θ. In the special case where prices are equal, the market share is no longer a function of θ, and the derivative of share with respect to θ equals zero. The cross-partials with respect to prices for $p_1 > p_2$ are

$$\frac{\partial^2 \tilde{s}_1}{\partial p_1 \partial p_2} = -\frac{1}{\theta^2}\sum_{j=1}^{n_1}(-1)^{j-1}\frac{{}_{n_1}C_j}{{}_{n_2+j}C_j}\cdot\frac{(n_2+j)^{-1}}{n_2+j-1}\left(1-\frac{\pi}{\theta}\right)^{n_2+j-2} \le 0\,,$$

$$\frac{\partial^2 \tilde{s}_1}{\partial \theta \partial p_1} = -\frac{\pi}{\theta^3}\sum_{j=1}^{n_1}(-1)^{j-1}\frac{{}_{n_1}C_j}{{}_{n_2+j}C_j}\cdot\frac{(n_2+j)^{-1}}{n_2+j-1}\left(1-\frac{\pi}{\theta}\right)^{n_2+j-2} \le 0\,,$$

$$\frac{\partial^2 \tilde{s}_1}{\partial \theta \partial p_2} = -\frac{\partial^2 \tilde{s}_1}{\partial \theta \partial p_1} \ge 0\,.$$

Effects of unit changes in the discrete parameters are more complicated:

Proposition 4. For the shares in Proposition 3:

$$\frac{\Delta \tilde{s}_1}{\Delta n_1} > 0,\ \frac{\Delta^2 \tilde{s}_1}{\Delta n_1^2} < 0,\ \frac{\Delta \tilde{s}_1}{\Delta n_2} < 0,\ \frac{\Delta^2 \tilde{s}_1}{\Delta n_2^2} < 0.$$

Similarly for $\tilde{s}_2$. ■

The Appendix presents the proof and magnitudes of the changes. Shares of each brand are increasing in own-variety at a decreasing rate and decreasing in other variety at a decreasing rate.

The cross effects are:

Proposition 5. For the shares in Proposition 3 and for $p_1 > p_2$:

$$\frac{\Delta \tilde{s}_1}{\Delta n_1 \Delta n_2} = \frac{n_2+1}{n_1+n_2+2}\left(1-\frac{\pi}{\theta}\right)\frac{\Delta \tilde{s}_1}{\Delta n_1} + \frac{n_1+1}{n_1+n_2+2}\frac{\Delta \tilde{s}_1}{\Delta n_2}.$$ ■

The proof is in the Appendix. The equation is a weighted average of two effects that determine the price-adjusted Coke share after both Coke and Pepsi variety have been incremented by 1.

The first effect—a weight times $\Delta\tilde{s}_1 / \Delta n_1$—is the positive effect of holding Pepsi variety fixed and increasing Coke variety, down-weighted by the extent to which the price of Coke exceeds that of Pepsi (1 – π/θ). The price weighting term, (1 - π/θ), precludes this change from being a pure marginal effect. If the price of Coke is very large this first effect may be negative, but for a moderate price differential, it will be positive. If there is no price differential, it will be positive with certainty.

The second effect—a weight times $\Delta\tilde{s}_1 / \Delta n_2$—is the pure negative effect of holding Coke variety fixed and increasing Pepsi variety. The weighting leads to a diminishing market share impact from increasing varieties. When there are many Pepsi varieties relative to Coke ($n_2 > n_1$), more weight $(n_2 + 1)$ is given to the (mostly) positive Coke demand effect and less weight $(n_1 + 1)$ is given to the purely negative Pepsi effect, so the cross-difference tends to be positive. When the opposite is true ($n_1 > n_2$), less weight is given to the (mostly) positive Coke effect and more weight is given to the purely negative Pepsi effect, so the cross-difference tends to be negative. (These effects hold in the absence of an outside good.)

The cross effects are clearer when the prices are equal so that the price differential is zero. Here, the formula reduces to

$$\frac{\Delta s_1}{\Delta n_1 \Delta n_2} = \frac{(n_2 + 1)}{(n_1 + n_2 + 2)} \frac{\Delta s_1}{\Delta n_1} + \frac{(n_1 + 1)}{(n_1 + n_2 + 2)} \frac{\Delta s_1}{\Delta n_2},$$

where $s_1(n_1, n_2) = \tilde{s}_1(n_1, n_2, p_1 = 0, p_2 = 0)$ as before. Now,

$$\frac{\Delta s_1}{\Delta n_1} = \frac{n_2}{(n_1 + n_2 + 1)(n_1 + n_2)} > 0,$$

$$\frac{\Delta s_1}{\Delta n_2} = \frac{-n_1}{(n_1 + n_2 + 1)(n_1 + n_2)} < 0.$$

Therefore,

$$\frac{\Delta s_1}{\Delta n_1 \Delta n_2} = \frac{n_2 - n_1}{(n_1 + n_2 + 2)(n_2 + n_1)},$$

which is positive for $n_2 > n_1$ and negative for $n_1 > n_2$. Thus, given small or moderate price differentials, the sign of the cross-partial is a function of the relative number of Coke and Pepsi varieties.

Different Supports

One way to capture a difference in how much consumers like one brand relative to another is to allow the uniform distributions for each brand to have different supports: $L_1 \in [0,\ \theta_1]$ and $L_2 \in [0,\ \theta_2]$. The difference in the upper bounds of the supports represent the extent to which consumers prefer one brand over the other given they can choose their most preferred variety from each. In other words, θ_1 and θ_2 are preference parameters.

Proposition 6. With different support the shares of Proposition 3 are

$$\tilde{s}_1(\theta_1,\theta_2) = \left(\frac{\theta_1}{\theta_2}\right)^{n_2} \sum_{j=1}^{n_1}(-1)^{j-1} \frac{{}_{n_1}C_j}{{}_{n_2+j}C_j}\left(1-\frac{\pi}{\theta_1}\right)^{n_2+j}, \text{ for } p_1-\theta_1 > p_2-\theta_2,$$

$$\tilde{s}_2(\theta_1,\theta_2) = \left(\frac{\theta_2}{\theta_1}\right)^{n_1} \sum_{j=1}^{n_2}(-1)^{j-1} \frac{{}_{n_2}C_j}{{}_{n_1+j}C_j}\left(1+\frac{\pi}{\theta_2}\right)^{n_1+j}, \text{ for } p_1-\theta_1 \le p_2-\theta_2. \ \blacksquare$$

The proof is similar to that in the symmetric case. If prices are zero or equal, these shares are:

$$s_1(n_1,n_2,\theta_1,\theta_2) = \left(\frac{\theta_1}{\theta_2}\right)^{n_2} \frac{n_1}{n_1+n_2}, \text{ for } \theta_2 > \theta_1,$$

$$s_2(n_1,n_2,\theta_1,\theta_2) = \left(\frac{\theta_2}{\theta_1}\right)^{n_1} \frac{n_2}{n_1+n_2}, \text{ for } \theta_1 \le \theta_2.$$

These equations show that Coke's share is increasing in θ_1 and decreasing in θ_2. All of the qualitative comparative statics results in the last section still apply with the obvious exception of those involving θ.

Non-negatively Valued Outside Good

In Proposition 3 (where we incorporated prices) we implicitly ignored a possible problem where a consumer would choose a Coke or Pepsi variety even though the net surplus for that good was negative. We could avoid that problem by having the uniform distribution start at a high enough level that a negative net surplus is impossible.[3] Instead, we introduce a non-negatively valued outside good with non-negative net surplus, $\omega \in [0, \min(\theta_1, \theta_2)]$. For simplicity, we assume that the net surplus of the outside good is non-random and the same for all consumers. We partition the domain of the joint distribution of $\ell_1 \in [-p_1,\ \theta_1 - p_1]$ and $\ell_2 \in [-p_2,\ \theta_2 - p_2]$ into four regions and outcomes. If the shares of Coke and Pepsi are s_1^* and s_2^*, respectively, then the following is true:

Proposition 7. The general share equations are,

$$s_1^* = \tilde{s}_1(\theta_1,\theta_2) + \left[1 - \left(\frac{p_1+\omega}{\theta_1}\right)^{n_1}\right]\left(\frac{p_2+\omega}{\theta_2}\right)^{n_2} - \left(\frac{\theta_1}{\theta_2}\right)^{n_2}\sum_{j=1}^{n_1}(-1)^{j-1}\frac{{}_{n_1}C_j}{{}_{n_2+j}C_j}\left(\frac{p_2+\omega}{\theta_1}\right)^{n_2+j}$$
$$-\frac{n_2}{n_1+1}\left(\frac{\theta_1}{\theta_2}\right)^{n_2}\sum_{j=1}^{n_1}(-1)^{j-1}\frac{{}_{n_1+1}C_j}{{}_{n_2-1+j}C_j}\left(\frac{p_2+\omega}{\theta_1}\right)^{n_2-1+j}\left[1-\left(\frac{p_1+\omega}{\theta_1}\right)^{n_1+1+j}\right],\quad p_1-\theta_1 > p_2-\theta_2,$$

$$s_1^* = \tilde{s}_1(\theta_1,\theta_2) - \left(\frac{p_1+\omega}{\theta_1}\right)^{n_1} + \left(\frac{\theta_2}{\theta_1}\right)^{n_1}\sum_{j=1}^{n_2}(-1)^{j-1}\frac{{}_{n_2}C_j}{{}_{n_1+j}C_j}\left(\frac{p_1+\omega}{\theta_2}\right)^{n_1+j}$$
$$+\frac{n_1}{n_2+1}\left(\frac{\theta_2}{\theta_1}\right)^{n_1}\sum_{j=1}^{n_2}(-1)^{j-1}\frac{{}_{n_2+1}C_j}{{}_{n_1-1+j}C_j}\left(\frac{p_1+\omega}{\theta_2}\right)^{n_1-1+j}\left[1-\left(\frac{p_2+\omega}{\theta_2}\right)^{n_2+1+j}\right],\quad p_1-\theta_1 \le p_2-\theta_2,$$

$$s_2^* = \tilde{s}_2(\theta_1,\theta_2) + \left[1 - \left(\frac{p_1+\omega}{\theta_1}\right)^{n_1}\right]\left(\frac{p_2+\omega}{\theta_2}\right)^{n_2} - \left(\frac{\theta_2}{\theta_1}\right)^{n_1}\sum_{j=1}^{n_2}(-1)^{j-1}\frac{{}_{n_2}C_j}{{}_{n_1+j}C_j}\left(\frac{p_1+\omega}{\theta_2}\right)^{n_1+j}$$
$$-\frac{n_1}{n_2+1}\left(\frac{\theta_2}{\theta_1}\right)^{n_1}\sum_{j=1}^{n_2}(-1)^{j-1}\frac{{}_{n_2+1}C_j}{{}_{n_1-1+j}C_j}\left(\frac{p_1+\omega}{\theta_2}\right)^{n_1-1+j}\left[1-\left(\frac{p_2+\omega}{\theta_2}\right)^{n_2+1+j}\right],\quad p_1-\theta_1 \le p_2-\theta_2.$$

[3] If the support for Coke is $[a, \theta_1]$ and that for Pepsi is $[a, \theta_2]$ with $\theta_1 > \theta_2$ and $p_1 = p_2$, then Pepsi's share is

$$s_2 = \left(\frac{\theta_2 - a}{\theta_1 - a}\right)^{n_1}\frac{n_2}{n_1+n_2}.$$

$$s_2^* = \tilde{s}_2(\theta_1,\theta_2) - \left(\frac{p_2+\omega}{\theta_2}\right)^{n_2} + \left(\frac{\theta_1}{\theta_2}\right)^{n_2} \sum_{j=1}^{n_1} (-1)^{j-1} \frac{{}_{n_1}C_j}{{}_{n_2+j}C_j} \left(\frac{p_2+\omega}{\theta_1}\right)^{n_2+j}$$
$$+ \frac{n_2}{n_1+1}\left(\frac{\theta_1}{\theta_2}\right)^{n_2} \sum_{j=1}^{n_1} (-1)^{j-1} \frac{{}_{n_1+1}C_j}{{}_{n_2-1+j}C_j} \left(\frac{p_2+\omega}{\theta_1}\right)^{n_2-1+j} \left[1-\left(\frac{p_1+\omega}{\theta_1}\right)^{n_1+1+j}\right], \; p_1-\theta_1 > p_2-\theta_2.$$

■

These are Hicksian (income compensated) demand functions, which we use below to calculate net welfare changes when varieties are changed and to calculate the socially optimal number of varieties. The demand function for the outside good, where the price of the outside good is included in ω is:

$$s_\omega^* = \left(\frac{p_1+\omega}{\theta_1}\right)^{n_1}\left(\frac{p_2+\omega}{\theta_2}\right)^{n_2}.$$

The shares of these three goods must sum to one: $s_1^* + s_2^* + s_\omega^* = 1$. Although derivates of the share equations are not presented, we present several numerical examples that show the effect of changes in the parameters on the shares of the three goods.

The area under the demand curves s_1^* and s_2^* can be used to calculate the effect of increasing varieties (Δn_1 or Δn_2) on total consumer surplus (*CS*). For instance, if we increase the varieties of Coke by one ($\Delta n_1 = 1$), demand for Coke increases so that demand for Pepsi and the outside good change, and the change (increase) in total CS can be represented as the difference of the areas under the compensated demand function for Coke (and above its price) before and after the change. That is, when $p_1 - \theta_1 > p_2 - \theta_2$,

$$\frac{\Delta CS(p_1-\theta_1 > p_2-\theta_2)}{\Delta n_1} = \int_{p_1}^{\theta_1-\omega} s_1^*(n_1+1, p_1-\theta_1 > p_2-\theta_2)dp_1 - \int_{p_1}^{\theta_1-\omega} s_1^*(n_1, p_1-\theta_1 > p_2-\theta_2)dp_1.$$

Notice the slight abuse of notation: the share, $s_1^*(n_1,\; p_1-\theta_1 > p_2-\theta_2)$, corresponds to Coke demand when $p_1-\theta_1 > p_2-\theta_2$. Because the demand functions are compensated, we need only calculate areas under the Coke function. We integrate with respect to price from the given price, p_1, to the upper bound, $\theta_1 - \omega$, because demand for Coke is zero if the price is higher (the outside good dominates for all consumers).

When $p_1 - \theta_1 \le p_2 - \theta_2$ the calculation is slightly different:

$$\frac{\Delta CS(p_1-\theta_1 \le p_2-\theta_2)}{\Delta n_1} = \int_{p_1}^{p_2-\theta_2+\theta_1} s_1^*(n_1+1,\ p_1-\theta_1 \le p_2-\theta_2)dp_1 - \int_{p_1}^{p_2-\theta_2+\theta_1} s_1^*(n_1,\ p_1-\theta_1 \le p_2-\theta_2)dp_1$$
$$+\int_{p_2-\theta_2+\theta_1}^{\theta_1-\omega} s_1^*(n_1+1,\ p_1-\theta_1 > p_2-\theta_2)dp_1 - \int_{p_2-\theta_2+\theta_1}^{\theta_1-\omega} s_1^*(n_1,\ p_1-\theta_1 > p_2-\theta_2)dp_1.$$

To calculate the change in total consumer surplus, we must use both parts of the demand function, s_1^*. The CS calculations are complicated because the demand equation for a brand differs depending on relative prices.

Similarly, the equation for s_2^* can be used to calculate the increase in total consumer demand when the varieties of Pepsi are increased by one. For example,

$$\frac{\Delta CS(p_1-\theta_1 \le p_2-\theta_2)}{\Delta n_2} = \int_{p_2}^{\theta_2-\omega} s_2^*(n_2+1,\ p_1-\theta_1 \le p_2-\theta_2)dp_2 - \int_{p_2}^{\theta_2-\omega} s_2^*(n_2,\ p_1-\theta_1 \le p_2-\theta_2)dp_2.$$

while the integrals are straight-forward to calculate, the resulting formulae are long and not presented here.

4. Coke and Pepsi Estimation

Our analytical results show that increasing the number of varieties of one brand can have complex effects on the demand curves and consumer welfare measures for both brands. To illustrate the role of variety on demand and on welfare, we estimate the simplest possible version of our model for Coke and Pepsi and an outside good in U.S. grocery stores. Previous estimates of the demand for Coke and Pepsi (Gasmi et al. 1992, Golan et al. 2000, Dhar et al. 2005, and Chan 2006) ignored or downplayed the role of variety.

Our order-statistic model has three parameters: The maximum value a consumer receives from a Coke variety, θ_1; the maximum value for a Pepsi variety, θ_2; and the surplus (value net of price) of the outside good, ω. Below, we discuss how various generalizations and variants of this model produce very similar results. We present this model as a plausible approximation of reality that allows us to simulate the demand and welfare effects of changes in variety.

Data

We use Information Resources Incorporated's (IRI) InfoScan® store-level scanner data for 1998 and 1998 to obtain 5,114 weekly observations for prices and quantities at 50 randomly chosen traditional grocery stores for each soft-drink variety (as determined by Universal Product Codes, UPCs).[4] The number of varieties that stores carry and the prices they charge vary across stores and over time within a store. These traditional grocery stores belong to 32 grocery chains. Some of the grocery chains are national giants such as Kroger, Albertsons, and Safeway, while others are relatively small, regional chains such as City Markets and Piggly Wiggly.

We restrict our analysis to 12-packs of 12 ounce cans. This package is the best-selling one within our data set, accounting for 46% of the total observations in the canned soft drink category. The number of varieties that each manufacturer produces is determined by the number of unique UPCs for the relevant package. Varieties differ by flavor, whether diet or regular, whether caffeinated or not, as well as how the products are packaged. Across all the stores in our sample, Coca-Cola has 27 varieties and Pepsi has 36 varieties.[5] Across all stores, the average annual number of varieties within each store is 10.86 (with a standard deviation of 2.76) for Coke and 9.08 (2.52) for Pepsi, the maximum number of varieties is 16 for both Coke and Pepsi, while the minimum number is 5 for Coke and 3 for Pepsi.

Each brand's store/week price is a quantity-weighted average obtained by dividing the total revenue in cents from all the products of the two firms by total volume in ounces. The price across varieties for a given brand is identical, but because of sales, the prices fluctuate over time (including sometimes within a week). The average price for

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[4] We have less than two full years of data for a few of the grocery stores that dropped out of the sample shortly before the end of the period. We also experimented with a panel of 100 stores for a single year (5,523 observations) and the results are very similar.

[5] Coke varieties include Coke, Diet Coke, Coke Classic, Caffeine Free Coke Classic, Caffeine-Free Diet Coke, Citra, Diet Cherry Coke, Diet Sprite, Fresca, Mello Yellow, Minute Maid, Diet Minute Maid Orange, Minute Maid Strawberry, Minute Maid Grape, Minute Maid Fruit Punch, Mr. Pibb, Sprite, Surge, and Tab. Pepsi varieties include Pepsi, Diet Pepsi, Caffeine-Free Pepsi, Caffeine-Free Diet Pepsi, Diet Minute Maid, Diet Mountain Dew, Mountain Dew Caffeine Free, Mountain Dew Citrus, Mug Root Beer, Mug Cream Soda, Josta, Diet Wild Cherry Pepsi, Pepsi One, Diet Pepsi Lemon Lime, Slice Strawberry, Slice Grape, Slice Mandarin Orange, Slice Lemon Lime, Slice Red, and Wild Cherry Pepsi.

Coke is 2.347¢ per ounce and that for Pepsi is 2.363¢ per ounce, with standard deviations of 0.514¢ and 0.473¢, respectively. The average price ratio of Pepsi to Coke across the stores is 1.02 with a standard deviation of 0.09. The correlation coefficient between Coke and Pepsi's price is 0.89. In other words, the prices of Coke and Pepsi are almost always equal.

There are a number of possibilities for the outside good. We use soda products of the same package and size products manufactured by firms other than Coca-Cola Co. and PepsiCo, including the stores' private label products. Data on the outside good other than share are not explicitly used in the estimation. We estimate a constant outside good net surplus ω, which is used to predict of shares of the outside good.

Share Estimates

We have observations over week *t* for store *i*. Our three-equation system of equations is

$$s_{1it}^{*} = s_{1}^{*}(n_{1it}, n_{2it}, p_{1it}, p_{2it}, \theta_1, \theta_2, \omega) + u_{1it},$$

$$s_{2it}^{*} = s_{2}^{*}(n_{1it}, n_{2it}, p_{1it}, p_{2it}, \theta_1, \theta_2, \omega) + u_{2it},$$

$$s_{1it}^{*} + s_{2it}^{*} = 1 - \left(\frac{p_{1it} + \omega}{\theta_1}\right)^{n_{1it}} \left(\frac{p_{2it} + \omega}{\theta_2}\right)^{n_{2it}},$$

where the shares are measured by volume. (The revenue shares are virtually identical as there is very little variation in the average absolute and relative prices of Coke and Pepsi.) The Coke and Pepsi shares are given by the shares in Proposition 7 with a random error term added to them. We estimate this three-equation system using a nonlinear, least squares, two-step method described by Davidson and MacKinnon (1993, p. 664). The nonlinear portion of the estimation is performed using the built-in constrained nonlinear estimation routine in MatLab.[6]

In our system of brand-share equations, we allow for the possibility that the prices and varieties are endogenously determined by using instrumental variables. Our

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[6] During each step of the estimation algorithm, we checked whether the constraints of the theoretical model were violated: $\omega \in [0, \min(\theta_1 - p_1, \theta_2 - p_2)]$. We observed no violation. Had we discovered violations, we would have used a constrained estimation algorithm.

instruments include cost shifters at the national level for the soft drink industry, the national share of each chain, and milk sales within each store (IRI). The cost shifters are the producer price index (PPI, from the Bureau of Labor Statistics) for high-fructose corn syrup, which is the main sweetener used in most soft drink; the PPI for aluminum, which is used to make cans; and the PPI for industrial electricity, and the national PPI for gasoline price interacted with city dummies, which is a proxy for variations in transportation costs across cities. We assume that national cost shifters are correlated with prices but are not correlated with underlying consumer preferences that vary from store to store or over time. The national shares of the chains are used as proxies for possible monopsony power. The within-store milk sales variable is a proxy for the size of the store. Presumably larger stores carry more varieties due to lower shelf-space costs; however, store size should not be explicitly correlated with the error terms of our share equations. We also include squared terms of these instruments as additional instruments because prices and varieties enter nonlinearly into the share equations (Davidson and MacKinnon, 1993).

In our first stage, we estimate the prices and number of varieties as linear functions of these instruments. The first-stage R^2 is 0.30 for Coke price, 0.30 for Pepsi price, 0.81 for the number of Coke varieties, and 0.86 for the number of Pepsi varieties.

Our order-statistics model estimates are extremely precise: $\theta_1 = 25.6181$ (with an asymptotic standard error of 0.0110), $\theta_2 = 25.2096$ (0.0103), and $\omega = 21.0219$ (0.0091). That is, all else equal, consumers slightly prefer Coke soft drinks to Pepsi's: $\theta_1 > \theta_2$.

This extremely terse model fits the data very well. Our model predicts the mean shares of Coke, Pepsi, and the outside good well. For example, Coke's average share in the data is 0.49, our model's average predicted share is 0.49, and the correlation coefficient between the actual and predicted share is 0.54. Similarly for Pepsi, the share is 0.26, the model's average predicted share is 0.27, and the correlation is 0.31. Finally, for the outside good, the corresponding numbers are 0.25, 0.24, and 0.16.

Comparison Models

For comparison, we also estimated the same model using multinomial logit and mixed logit. In the multinomial logit model, the shares of the goods are a function of the prices, the number of varieties, a Coca-Cola dummy, and store dummies, and all variables are treated as exogenous. The multinomial logit fits the data only slightly better than our model. For example, the Coke share correlation is 0.60 for the multinomial logit and 0.54 for our model. However, the multinomial logit does not use instruments and hence is unlikely to produce consistent estimates.

The specification of the mixed logit model includes the variables in the multinomial logit and randomly drawn errors from a normal distribution with zero mean. The dependent variables are the logarithm share of Coca-Cola minus the log outside share and the log Pepsi share minus the log outside share. The mixed logit does use instruments and fits well in the sense that it produces nearly perfect correlations with the shares. However, its estimated parameters are nonsensical. For example, the estimated coefficient for the number of varieties is negative.

Robustness Checks

We estimated three variations of our basic model with an outside good to check for the robustness of our results. First, we eliminated the main variety for each brand (Coke and Pepsi), which have very large market shares, and re-estimated the model. The estimated coefficients and the fit are virtually unaffected.

Second, we modified our model so that the three coefficients were linear functions of the average household income and the average number of family members in a household at a store. Doing so reduced the correlation coefficients (the fit of the model), and the coefficients on the demographic variables were not statistically significantly different from zero. The estimated model was very close to our three-parameter version.

Third, using a sample of 106 stores over a single year (5,523 observations), we estimated a model that allowed the three parameters to vary over stores. That is, $\theta_{1i} = \theta_1 + \delta_{1i}$; $\theta_{2i} = \theta_2 + \delta_{2i}$; and $\omega_i = \omega + \delta_{\omega i}$, where the δ's are estimated coefficients for each store. Presumably, the store dummies capture differences in all (average) demographic and other variables that vary across stores. The results of this flexible model were similar to the three-parameter model. The basic parameter estimates were similar: $\theta_1 = 25.5723$, $\theta_2 = 25.3866$, and $\omega = 21.4642$ (compare to our reported estimates of 25.6181, 25.2096, 21.0219, respectively). Also, the estimated δ's were relatively small in magnitude (average 0.1287 and standard deviation 0.2590). For example, 98% of the estimated θ_{1i}

and θ_{2i} were between 25 and 26, with the three exceptions being for a few stores with very high Pepsi prices and low Pepsi varieties. Also, in all cases Coke was preferred to Pepsi at the store level ($\theta_{1i} > \theta_{2i}$). Thus, we use our simple three-parameter model in the following simulations as it simplifies the analysis.

5. Coke and Pepsi Calculations and Simulations

Using our estimated values for θ_1, θ_2, and ω, we can calculate demand curves and welfare measures. We can also simulate comparative statics results.

Unless otherwise stated, the following calculations are made using the sample average price of p_1 = 2.35¢ per ounce for Coke and p_2 = 2.36¢ per ounce for Pepsi, and the sample average number of varieties of n_1 = 11 for Coke and n_2 = 9 for Pepsi.

Demand Curves

At the sample average prices, the own price demand elasticity for Pepsi is -1.5. That is, a 1% increase in the price of Pepsi holding the price of Coke and the number of varieties fixed lowers the quantity of Pepsi demanded by 1.5%. The price demand elasticity for Coke, -0.9, is less elastic.

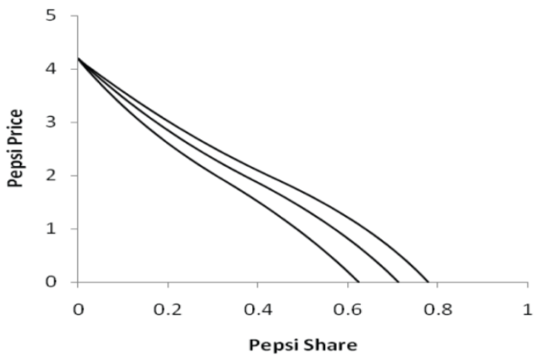
As the results for Coke and Pepsi are qualitatively the same, we will present graphical results for changes in only Pepsi's price and number of varieties. By varying the number of varieties, we can show how the demand curves shift, thereby illustrating our analytic partial and cross-partial derivative results.

Panel a of Figure 1 shows three Pepsi demand curves. In each demand curve, we fix the price and varieties of Coke at their sample averages, p_1 = 2.35¢ per ounce and n_1 = 11. Moving from left to right we increase the number of Pepsi varieties in increments of two from n_2 = 7 to n_2 = 9—the observed average number—to n_2 = 11. The figure illustrates that, as the number of varieties of Pepsi increases, Pepsi's demand curve rotates around the price-axis intercept, becoming flatter. At the sample mean prices, one extra Pepsi variety—n_2 goes from 9 to 10—increases the Pepsi's share by 0.0233, reduces Coke's by 0.0099, and reduces the share of the outside good by 0.0134 (= 0.0233 – 0.0099).

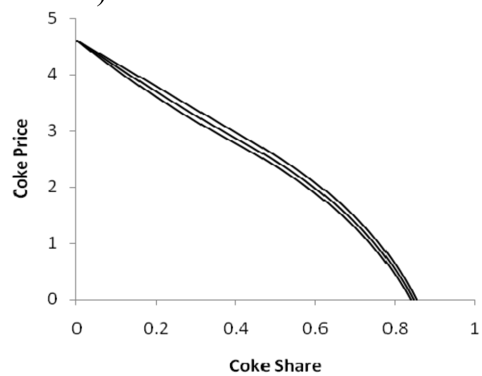
We illustrate the effect of the increase in the number of Pepsi varieties on the Coke demand curve in panel b of Figure 1. As the number of Pepsi varieties increases

Figure 1
Effect on Pepsi's Demand Curve as the Number of Varieties Change

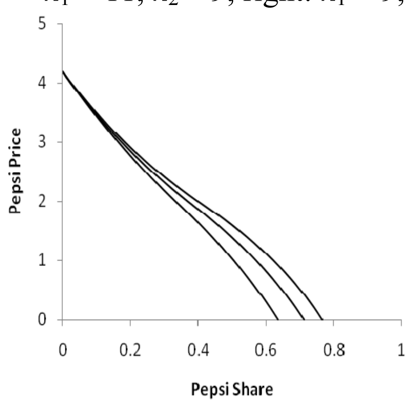
(a) Shift in Pepsi demand curve as the number of Pepsi varieties increases from $n_2 = 7$ to 9 to 11 (left to right)



(b) Shift in Coke demand curves as the number of Pepsi varieties increases from $n_2 = 7$ to 9 to 11 (right to left)



(c) Pepsi demand curves: Cross-partial effect (left: $n_1 = 13$, $n_2 = 11$; middle: $n_1 = 11$, $n_2 = 9$; right: $n_1 = 9$, $n_2 = 7$)



Note: Unless other stated, these simulations are based on $n_1 = 11$, $n_2 = 9$, $p_1 =$ 2.347¢ per ounce, $p_2 = 2.363$¢ per ounce, $\theta_1 = 25.6181$, $\theta_2 = 25.2096$, and $\omega =$ 21.0219.

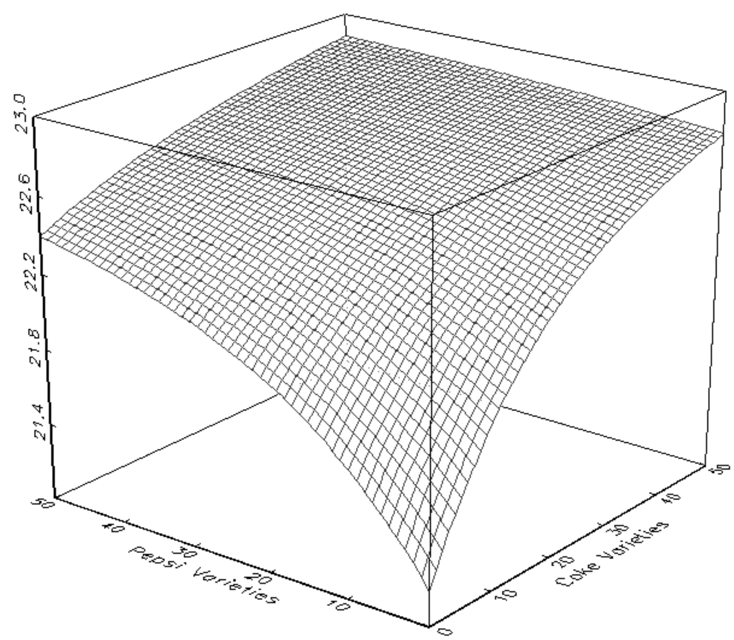
from $n_2 = 7$ to 9 to 11, the Coke demand curve rotates in around the price-axis intercept. Thus, an increase in the number of Pepsi varieties causes the Coke demand curve to become steeper. As these graphs illustrate, a change in the number of Pepsi varieties has a larger (own) effect on Pepsi's demand curve than its (cross) effect on Coke's demand curve. Similarly, changes in the number of Coke varieties have a larger effect on its demand curve than on Pepsi's.

Panel c of Figure 1 illustrates cross-partial effects for Pepsi demand curves. The central demand curve is evaluated at the sample averages where $n_1 = 11$ and $n_2 = 9$. The one to its left has two fewer varieties of each brand, $n_1 = 9$ and $n_2 = 7$, while the one to its right has two more varieties, $n_1 = 13$ and $n_2 = 11$. When both brands have more varieties, fewer consumers buy the outside good, so the demand curves for both brands shift to the right. The effect is larger for Pepsi than for Coke because there are more Coke varieties, as our analytic results indicate.

Consumer Surplus

Figure 2 shows that consumer surplus is increasing at a decreasing rate in the number of varieties of both goods. The figure is slightly asymmetric because of the

Figure 2
Total Consumer Surplus as a Function of the Number of Coke and Pepsi Varieties
(The number of varieties range from 0 to 50)



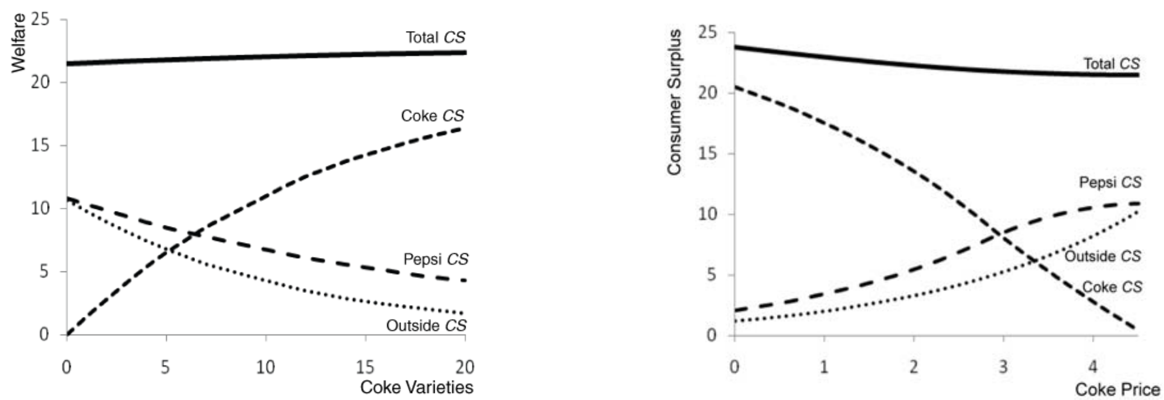
Note: In these simulations, $p_1 = 2.347$¢ per ounce, $p_2 = 2.363$¢ per ounce, $\theta_1 = 25.6181$, and $\theta_2 = 25.2096$, $\omega = 21.0219$ (so that when $n_1 = n_2 = 0$, total consumer surplus equals 21.0219).

preference for Coke ($\theta_1 > \theta_2$).

Consumer surplus varies with the number of varieties and price, as Figure 3 demonstrates. In panel a, as the number of varieties of Coke increases holding the number of varieties of Pepsi and prices fixed, the consumer surplus of Coke rises, while that of Pepsi and the outside good fall. (The curves interpolate and smooth the discrete changes in the varieties of Coke.) Consequently, total consumer surplus rises, but only slightly as the gain to Coke barely exceeds the combined losses from Pepsi and the outside good.

Figure 3
Variation in Consumer Surplus with the Number of Coke Varieties or Price

(a) Consumer Surplus and Coke's Varieties (b) Consumer Surplus and Coke's Price

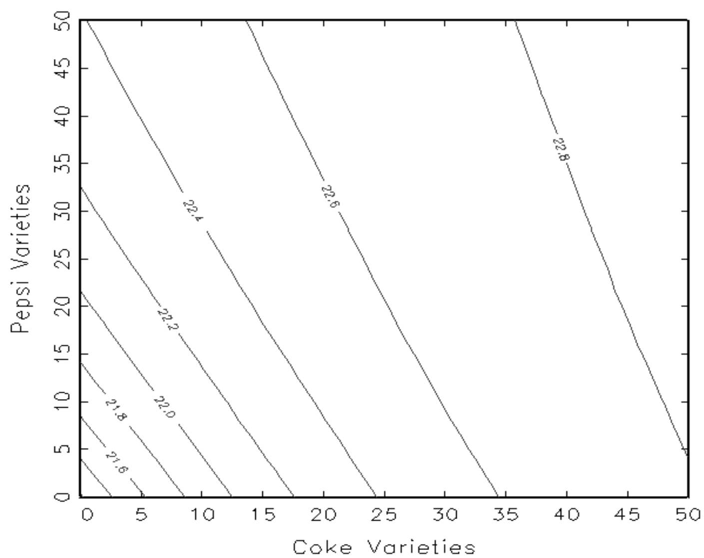


Note: In these simulations, n_1 = 11 in (b), n_2 = 9, Coke's price = 2.347¢ per ounce in (a), Pepsi's price = 2.363¢ per ounce, θ_1 = 25.6181, θ_2 = 25.2096, and ω = 21.0219.

As the price of Coke increases, holding the price of Pepsi constant, the consumer surplus from Coke falls, while the consumer surplus from Pepsi and that from the outside good rise, as panel b of Figure 4 illustrates. As the price gets very large, total consumer surplus levels off and there is little decrease in total surplus because virtually all consumers have switched to the other goods.

Figure 4 shows "iso-welfare" curves, where each curve holds consumer surplus constant and the number of varieties of each brand vary (treating the number of varieties as a continuous variable). These are curves are horizontal slices of the three-dimensional surface in Figure 2. These curves are virtually straight lines with slope less than -1. That is, consumers are willing to trade slightly more than one Pepsi variety for a single Coke variety to keep consumer surplus constant. This slight deviation from -1 is the result of a slight preference for Coke: $\theta_1 - p_1 > \theta_2 - p_2$. Indeed, the slope becomes slightly more negative as the number of varieties increase, so that the iso-welfare lines are not parallel. That is as varieties increase, it takes a greater increase in Pepsi varieties to offset the loss of one Coke variety. This same effect appears in Figure 2. Total surplus is increasing as a decreating rate in variety, but the rate decreases more slowly for Coke than for Pepsi. The cross-partial analysis is consistent with these figures: If we increase by one the number of varieties of both goods, Coke's share rises by 0.0191, which is more than Pepsi's share increases, 0.0105.

Figure 4
Iso-Welfare Curves



6. Optimal Varieties

We can use our estimated model to investigate whether grocery stores carry the socially optimal number of varieties. Our welfare analysis looks only at the grocery store level; it does not consider welfare effects at the manufacturing level.

A grocery stores' profit is

$$\Pi = (p_1 - m)s_1^* Z + (p_2 - m)s_2^* Z + (p_\varpi - m_\varpi)s_\varpi^* Z - (cn_1 + kn_1^2) - (cn_2 + kn_2^2)$$

where m is the common wholesale price for Coke and Pepsi; m_ϖ is the wholesale price of other soft drinks; Z = total ounces of all soft-drinks including the outside good; s_1^* and s_2^*

are functions of all the prices, varieties, marginal costs, and estimated parameters θ_1, θ_2, and ω; and the share of the outside good is $s_\omega^* = 1 - s_1^* - s_2^*$.

In our simulations, we assume that the typical grocery store carries 11 varieties of Coke and 9 of Pepsi because doing so maximizes its profit. We also assume that grocery stores are price takers.[7] That is, their only control variables are the numbers of varieties of Coke and Pepsi that they carry. Presumably the store's costs vary with the number of varieties they carry for a given brand due to shelf-space opportunity costs as well as storage, accounting, and other costs. The store incurs shelf-space costs, inventory costs, label costs, and other expenses from carrying an additional variety. It is possible that some of these costs are offset by slotting, a fee that manufacturers pay when a grocery store agrees to carry one more variety. We assume that the (net) cost with respect to varieties is quadratic: $cn_i + kn_i^2$.

The store's first-order conditions to maximize profit with respect to a discrete (partial) change in the number of varieties of each brand are:

$$\frac{\Delta\Pi}{\Delta n_1} = (p_1 - m)\frac{\Delta s_1^*}{\Delta n_1}Z + (p_2 - m)\frac{\Delta s_2^*}{\Delta n_1}Z + (p_\omega - m_\omega)\frac{\Delta s_\omega^*}{\Delta n_1}Z - c - k(2n_1 + 1) = 0$$

$$\frac{\Delta\Pi}{\Delta n_2} = (p_1 - m)\frac{\Delta s_1^*}{\Delta n_2}Z + (p_2 - m)\frac{\Delta s_2^*}{\Delta n_2}Z + (p_\omega - m_\omega)\frac{\Delta s_\omega^*}{\Delta n_2}Z - c - k(2n_2 + 1) = 0$$

That is, the firm sets the number of varieties so that the marginal profit from the last variety (the first three terms) equals the marginal cost of one more variety (the last two terms). Given what we know from our analytic results, our estimation, and the summary statistics, if we also know the marginal costs, we can solve this system of two equations for the two unknowns, *c* and *k*.

We can use our "estimates" of *c* and *k* to compare the optimal number of varieties under profit maximization to three possible outcomes determined by a social planner who

[7] As many types of retailers (e.g., grocery stores, drug stores, warehouse stores, and restaurants) sell soft drinks, it is plausible that grocery stores are price takers. The average coefficient of variation of the price across stores is relatively small despite frequent sales: 0.13 for Coke and 0.11 for Pepsi, and 0.18 for the ratio of the Coke to Pepsi price (with mean of 1.01). There are only small differences in prices regionally: The highest average Coke price is 0.026 in the North East compared to the lowest of 0.021 in the South. There is little variation in the average Coke and Pepsi prices across stores within a city.

is interested in maximizing welfare, defined as consumer surplus plus profit (for soft drinks):

(a) the *social optimum,* where a social planner maximizes social welfare by setting the Coke, Pepsi, and outside good prices and Coke and Pepsi varieties;

(b) the second-best *varieties-only optimum*, where the planner sets only the number of varieties for Coke and Pepsi;

(c) the second-best *prices-only optimum*, where the planner sets the price of Coke, Pepsi, and the outside good.

These two second-best approaches provide a means of (approximately) decomposing the welfare gain from the social optimum approach, so that we can determine if most of the gain comes from controlling prices or varieties.

We solve for the social optimum by setting prices equal to their marginal cost, and then choosing the numbers of varieties to maximize welfare. The planner sets the marginal cost of an extra variety equal to the marginal welfare from one more variety, where the marginal welfare is the marginal profit plus the marginal consumer surplus.[8] That is, the planner's marginal benefit contains one more term, the marginal consumer surplus, than does the firm's marginal benefit. We assume that the planner subsidizes the firm so that it does not shut down (otherwise it would make a negative profit because it makes nothing per unit sold because price equals marginal cost and it incurs the variety cost).

According to one major chain that we consulted, the marginal cost is about 30% to 40% of price (depending on the frequency of sales). In Table 1, we assume that m is 1¢ per ounce (about 42% of the observed price) and m_ω is 1.5¢ per ounce.[9] We find that, if we vary the marginal cost measures proportionately, the following qualitative results hold.

The first column of Table 1 reports sample averages, which we call the profit-maximizing solution. We want to compare this profit-maximizing solution to the social

[8] As a practical matter, we do an exhaustive comparison of welfare for all plausible pairs of numbers of varieties to find the social optimum.

[9] We set the wholesale price of the outside good higher than that of Coke and Pepsi because otherwise, we often get a corner solution.

Table 1
Profit Maximization vs. Social Optima

		Planner Sets		
	Profit Max	*Prices & Varieties*	*Varieties Only*	*Prices Only*
n_1	11	26	27	11
n_2	9	11	11	9
p_1	2.35¢	1¢	2.35¢	1¢
p_2	2.36¢	1¢	2.36¢	1¢
p_ω	2.21¢	1.5¢	2.21¢	1.5¢
Profit	$2,447	-$797	$2,086	-$273
Variety Cost	$273	$797	$840	$273
CS	$48,729	$52,662	$49,719	$51,510
Welfare	$51,176	$51,825	$51,805	$51,237

Notes: We set m = 1¢, m_ω = 1.5¢, so that c = 666.89¢, k = 69.05¢

optimum (second column) and to the second-best varieties-only optimum, where the planner sets only varieties (third column) and the prices-only optimum (fourth column).

The social optimum prices are, of course, lower than in the unregulated case. When we switch from the profit-maximizing solution to the social optimum, the numbers of varieties increase from 11 to 26 for Coke, n_1, and from 9 to 11 for Pepsi, n_2. In the varieties-only optimum, n_2 increases by the same as in the social optimum, but n_1 increases more than in the social optimum to 27. We conclude that society would be better off if a typical grocery carried substantially more Coke varieties and slightly more of Pepsi varieties.

In our typical store, our weekly consumer surplus, *CS*, is $48,729 without regulation, $52,662 (8.1% higher than without regulation) in the social optimum, $49,719 (2.0% higher) in the varieties-only optimum, and $51,510 (5.7% higher) in the prices-only optimum. Thus, most of the gain in consumer surplus is due to regulating price rather than the number of varieties.

Welfare is $51,176 without regulation, $51,825 (1.3% higher than without regulation) in the social optimum, $51,805 (1.2% higher) in the varieties-only optimum, and $51,237 (0.1% higher) in the prices-only optimum. Thus, in contrast to the *CS* results, most of the welfare gain is due to regulating varieties rather than prices.

Varying the marginal costs, we find qualitatively similar effects, but the size of the price effect rises when the marginal cost falls. We conclude that consumers and society as a whole would benefit from more varieties of Coke and Pepsi. However, even with extreme marginal cost values, regulating only varieties raises consumer surplus and welfare by only relatively small percentages.

7. Summary and Conclusions

We examine markets in which duopoly brands sell many varieties. As the number of varieties of one brand increases holding prices the varieties of the other brand fixed, consumers are more likely to buy that brand, as more consumers will find a variety that they prefer to those of the other brand and to the outside good.

Rather than imposing an explicit functional form on utility or demand, we derive consumers' demand functions using a new model of consumer demand based on order statistics where consumers' valuations of varieties are distributed independently uniform. Consumer choices are made based on preference ordering over the net consumer surpluses of the two brands and an outside good.

We derive explicit demand functions for each of the duopoly brands. We also derive analytic partial derivative results that show how changes in price and varieties affect demand and consumer surplus.

We estimate this model for Coke, Pepsi, and an outside good based on U.S. grocery store data. The model fits the data very well and predicts within-sample shares better than do multinomial logit and mixed logit models.

Using these estimates, we simulate the demand and welfare comparative statics results for changes in prices, varieties, and the value of the outside good. We also examine whether stores carry the socially optimal number of varieties conditional on an assumption that the store's cost function over varieties is quadratic. According to our simulations, grocery stores carry substantially too few varieties, but the consumer surplus and welfare effects are relatively small.

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Appendix

Proof of Proposition 1.

The marginal distributions of the maxima are $n_1\theta^{-n_1}L_1^{n_1-1}dL_1$ and $n_2\theta^{-n_2}L_2^{n_2-1}dL_2$, and the cumulations below *c* of each maxima are $\theta^{-n_1}c_1^{n_1}$ and $\theta^{-n_2}c_2^{n_2}$. The joint distribution of the maximal net surplus of Coke and Pepsi is:

$$f_L(L_1,L_2)=n_1n_2\theta^{-(n_1+n_2)}L_1^{n_1-1}L_2^{n_2-1},$$

with cumulations below $(c_1,\ c_2)$ as $\theta^{-(n_1+n_2)}c_1^{n_1}c_2^{n_2}$. The distribution of the difference of the maxima, $D_0=L_1-L_2\in[-\theta,\ \theta]$, has two parts:

$$f_{D_0}(D_0)=\left\{\begin{array}{ll}\int_0^{\theta-D_0} f_L(D_0+L_2,L_2)dL_2 & D_0\in(0,\ \theta] \\ \int_0^{\theta+D_0} f_L(L_1,L_1-D_0)dL_1 & D_0\in[-\theta,\ 0]\end{array}\right\}.$$

The upper-bound of the first integral maps $D_0\in[0,\ \theta]$ through $L_2=\theta-D_0$ to $L_2\in[0,\ \theta]$, and the upper-bound of the second maps $D_0\in[-\theta,\ 0]$ through $L_1=\theta+D_0$ to $L_1\in[0,\ \theta]$. After repeated integration by parts:

$$f_{D_0}(D_0)=\left\{\begin{array}{ll}\frac{n_2}{\theta^{m+n}}\sum_{j=1}^{m}(-1)^{j-1}\frac{{}_{n_1}C_j}{{}_{n_2-1+j}C_j}(L_2+D_0)^{n_1-j}(L_2)^{n_2-1+j}\Big|_0^{\theta-D_0} & D_0\in(0,\ \theta] \\ \frac{n_1}{\theta^{m+n}}\sum_{j=1}^{n}(-1)^{j-1}\frac{{}_{n_2}C_j}{{}_{n_1-1+j}C_j}(L_1)^{n_1-j}(L_1-D_0)^{n_2-1+j}\Big|_0^{\theta+D_0} & D_0\in[-\theta,\ 0]\end{array}\right\}$$

Evaluating the limits of integration:

$$f_{D_0}(D_0)=\left\{\begin{array}{ll}\frac{n_2}{\theta}\sum_{j=1}^{m}(-1)^{j-1}\frac{{}_{n_1}C_j}{{}_{n_2-1+j}C_j}\left(1-\frac{D_0}{\theta}\right)^{n_2-1+j}, & D_0\in(0,\ \theta] \\ \frac{n_1}{\theta}\sum_{j=1}^{n}(-1)^{j-1}\frac{{}_{n_2}C_j}{{}_{n_1-1+j}C_j}\left(1+\frac{D_0}{\theta}\right)^{n_1-1+j}, & D_0\in[-\theta,\ 0]\end{array}\right\}.$$

Evaluating cumulations above and below zero the results of Proposition 1 follow. ■

Proof of Proposition 3.

The joint distribution of the maxima adjusted for price is:

$$f_\ell(\ell_1,\ell_2) = n_1 n_2 \theta^{-(n_1+n_2)}(\ell_1+p_1)^{n_1-1}(\ell_2+p_2)^{n_2-1},\ \ell_g \in [-p_g, \theta-p_g],\ g=1,2.$$

The distribution of the difference $D = \ell_1 - \ell_2 = D_0 - \pi$ is:

$$f_D(D) = \left\{ \begin{array}{ll} \int_{-p_2}^{\theta-p_1-D} f_\ell(\ell_2+D,\ell_2)d\ell_2, & D\in(-\pi,\ \theta-\pi],\ p_1>p_2 \\ \int_{-p_1}^{\theta-p_2+D} f_\ell(\ell_1,\ell_1-D)d\ell_1, & D\in[-\theta-\pi,\ -\pi],\ p_1\le p_2 \end{array} \right\}.$$

The upper-bound of the first integral maps $D\in[-\pi,\ \theta-\pi]$ through $\ell_2 = \theta - p_1 - D$ to $\ell_2 \in [-p_2,\ \theta-p_2]$, and the upper-bound of the second maps $D\in[-\theta-\pi,\ -\pi]$ through $\ell_1 = \theta - p_2 + D$ to $\ell_1 \in [-p_1,\ \theta-p_1]$. After repeated integration by parts:

$$f_D(D) = \left\{ \begin{array}{ll} \frac{n_2}{\theta^{m+n}}\sum_{j=1}^{m}(-1)^{j-1}\frac{{}_{n_1}C_j}{{}_{n_2-1+j}C_j}(\ell_2+D+p_1)^{n_1-j}(\ell_2+p_2)^{n_2-1+j}\Big|_0^{\theta-p_1-D} & D\in(0,\ \theta-\pi] \\ \frac{n_1}{\theta^{m+n}}\sum_{j=1}^{n}(-1)^{j-1}\frac{{}_{n_2}C_j}{{}_{n_1-1+j}C_j}(\ell_1+p_1)^{n_1-j}(\ell_1-D+p_2)^{n_2-1+j}\Big|_0^{\theta-p_2+D} & D\in[-\theta-\pi,\ 0] \end{array} \right\}.$$

Evaluating the limits of integration:

$$f_D(D) = \left\{ \begin{array}{ll} \frac{n_2}{\theta}\sum_{j=1}^{n_1}(-1)^{j-1}\frac{{}_{n_1}C_j}{{}_{n_2-1+j}C_j}\left(1-\frac{\pi-D}{\theta}\right)^{n_2-1+j} & D\in(-\pi,\ \theta-\pi],\ p_1>p_2 \\ \frac{n_1}{\theta}\sum_{j=1}^{n_2}(-1)^{j-1}\frac{{}_{n_2}C_j}{{}_{n_1-1+j}C_j}\left(1+\frac{\pi+D}{\theta}\right)^{n_1-1+j} & D\in[-\theta-\pi,\ -\pi],\ p_1\le p_2 \end{array} \right\}.$$

As one might expect, this distribution is the same as the distribution of D_0 displaced by the price differential. Evaluating cumulations above and below zero the results of Proposition 3 follow. ■

Proof of Proposition 4.

First consider the case where $p_1 > p_2$. If we increase n_1 by one in Proposition 3,

$$\tilde{s}_1(n_1+1) = \sum_{j=1}^{n_1+1}(-1)^{j-1}\frac{{}_{n_1+1}C_j}{{}_{n_2+j}C_j}\left(1-\frac{\pi}{\theta}\right)^{n_2+j} = \sum_{j=1}^{n_1+1}(-1)^{j-1}\frac{{}_{n_1}C_j}{{}_{n_2+j}C_j}\left(1-\frac{\pi}{\theta}\right)^{n_2+j}\left(\frac{n_1+1}{n_1+1-j}\right)$$
$$> \sum_{j=1}^{n_1+1}(-1)^{j-1}\frac{{}_{n_1}C_j}{{}_{n_2+j}C_j}\left(1-\frac{\pi}{\theta}\right)^{n_2+j} = \sum_{j=1}^{n_1}(-1)^{j-1}\frac{{}_{n_1}C_j}{{}_{n_2+j}C_j}\left(1-\frac{\pi}{\theta}\right)^{n_2+j} = \tilde{s}_1.$$

This result hinges follows because $(n_1 + 1)/(n_1 + 1 - j) > 1$ and that the $(n_1 + 1)^{th}$ term in the sum of the second to last line is zero, because ${}_{n_1}C_{n_1+1} = 0$, Thus we have:

$$\frac{\Delta\tilde{s}_1}{\Delta n_1} = \sum_{j=1}^{n_1}(-1)^{j-1}\frac{{}_{n_1}C_j}{{}_{n_2+j}C_j}\left(1-\frac{\pi}{\theta}\right)^{n_2+j}\left(\frac{j}{n_1+1-j}\right) > 0.$$

Taking the difference of the differences shows $\frac{\Delta^2\tilde{s}_1}{\Delta n_1^2} < 0$. Increasing n_2 in Proposition 3,

$$\tilde{s}_1(n_2+1) = \sum_{j=1}^{n_1}(-1)^{j-1}\frac{{}_{n_1}C_j}{{}_{n_2+1+j}C_j}\left(1-\frac{\pi}{\theta}\right)^{n_2+1+j}$$
$$= \sum_{j=1}^{n_1}(-1)^{j-1}\frac{{}_{n_1}C_j}{{}_{n_2+j}C_j}\left(1-\frac{\pi}{\theta}\right)^{n_2+j}\left(\frac{n_2+1}{n_2+1+j}\cdot\left(1-\frac{\pi}{\theta}\right)\right)$$
$$< \sum_{j=1}^{n_1}(-1)^{j-1}\frac{{}_{n_1}C_j}{{}_{n_2+j}C_j}\left(1-\frac{\pi}{\theta}\right)^{n_2+j} = \tilde{s}_1(n_2).$$

This result follows because $(n_2 + 1)/(n_2 + 1 + j) < 1$ and $(1 - \pi/\theta) \leq 1$, which decreases each term in the first equation of(1), thus we have for $p_1 > p_2$:

$$\frac{\Delta\tilde{s}_1(n_2)}{\Delta n_2} = -\sum_{j=1}^{n_1}(-1)^{j-1}\frac{{}_{n_1}C_j}{{}_{n_2+j}C_j}\left(1-\frac{\pi}{\theta}\right)^{n_2+j}\left(1-\frac{n_2+1}{n_2+1+j}\cdot\left(1-\frac{\pi}{\theta}\right)\right) < 0$$

Taking the difference of the differences it is easy to show $\frac{\Delta^2\tilde{s}_1}{\Delta n_2^2} < 0$. Similar results for the case $p_1 < p_2$ can be derived. Thus completing proof of Proposition 4. ■

Proof of Proposition 5.

To understand the cross-difference (cross-partial difference) with respect to n_1 and n_2 it is useful to restate the shares in terms of the Gaussian hypergeometric function (Gauss 1813 or Pachhammer 1870):

$$F(a,\ b;\ c;\ z) = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j\, j!} z^j,$$

where $(a)_j = a(a+1)...(a+j-1)$ is the Pochhammer rising factorial. When the first argument is negative, the hypergeometric sum is finite. As a result, the probability in Proposition 3 can be restated as:

$$\tilde{s}_1(n_1, n_2, p_1, p_2) = \left(1 - \frac{\pi}{\theta}\right)^{n_2} \left[1 - F(-n_1,\ 1;\ n_2+1;\ 1-\pi/\theta)\right].$$

Rearranging:

$$F(-n_1,\ 1;\ n_2+1;\ 1-\pi/\theta) = 1 - \left(1 - \frac{\pi}{\theta}\right)^{-n_2} \tilde{s}_1.$$

Incrementing n_1 and n_2:

$$F(-n_1-1,\ 1;\ n_2+1;\ 1-\pi/\theta) = 1 - \left(1 - \frac{\pi}{\theta}\right)^{-n_2} \tilde{s}_1(n_1+1),$$

$$F(-n_1,\ 1;\ n_2+2;\ 1-\pi/\theta) = 1 - \left(1 - \frac{\pi}{\theta}\right)^{-n_2-1} \tilde{s}_1(n_2+1),$$

$$F(-n_1-1,\ 1;\ n_2+2;\ 1-\pi/\theta) = 1 - \left(1 - \frac{\pi}{\theta}\right)^{-n_2-1} \tilde{s}_1(n_1+1, n_2+1).$$

Substituting these into Equation 15.2.17 of Abramowitz and Stegun (1972) yields:

$$\tilde{s}_1(n_1+1, n_2+1) = \frac{n_2+1}{n_1+n_2+2}\left(1 - \frac{\pi}{\theta}\right)\tilde{s}_1(n_1+1) + \frac{n_1+1}{n_1+n_2+2}\tilde{s}_1(n_2+1).$$

Subtracting $\tilde{s}_1$ yields the result of Proposition 5. ■

Proof of Proposition 7.

Consider the following regions in two-dimensional Cartesian space.

Region	*Market Share*
$R_{\omega\omega} = \{(\ell_1,\ \ell_2):\ \ell_1 \in [-p_1,\ \omega),\ \ell_2 \in [-p_2,\ \omega)\}$	$s_1^* = s_2^* = 0$
$R_{1\omega} = \{(\ell_1,\ \ell_2):\ \ell_1 \in [\omega,\ \theta_1 - p_1],\ \ell_2 \in [-p_2,\ \omega)\}$	$s_1^* \in [0,\ 1]$; $s_2^* = 0$
$R_{\omega 2} = \{(\ell_1,\ \ell_2):\ \ell_1 \in [-p_1,\ \omega),\ \ell_2 \in [\omega,\ \theta_2 - p_2]\}$	$s_1^* = 0$; $s_2^* \in [0,\ 1]$
$R_{12} = \{(\ell_1,\ \ell_2):\ \ell_1 \in [\omega,\ \theta_1 - p_1],\ \ell_2 \in [\omega,\ \theta_2 - p_2]\}$	$s_1^* \in [0,\ 1]$; $s_2^* \in [0,\ 1]$

Given that some consumers purchase the outside good instead of a Coke or Pepsi variety, $s_1^* + s_2^* \le 1$. Coke and Pepsi shares equal the probabilities:

$$(*)\quad s_1^*(n_1,n_2,p_1,p_2,\theta_1,\theta_2) = \begin{Bmatrix} \Pr(\ell_1 > \ell_2 \cap R_{12}) + \Pr(R_{1\omega}) & p_1 - \theta_1 > p_2 - \theta_2 \\ 1 - \Pr(\ell_1 \le \ell_2 \cap R_{12}) - \Pr(R_{\omega 2}) - \Pr(R_{\omega\omega}) & p_1 - \theta_1 \le p_2 - \theta_2 \end{Bmatrix},$$

$$(**)\quad s_2^*(n_1,n_2,p_1,p_2,\theta_1,\theta_2) = \begin{Bmatrix} \Pr(\ell_1 \le \ell_2 \cap R_{12}) + \Pr(R_{\omega 2}) & p_1 - \theta_1 \le p_2 - \theta_2 \\ 1 - \Pr(\ell_1 > \ell_2 \cap R_{12}) - \Pr(R_{1\omega}) - \Pr(R_{\omega\omega}) & p_1 - \theta_1 > p_2 - \theta_2 \end{Bmatrix}.$$

The probability masses on $R_{1\omega}$, $R_{\omega 2}$, and $R_{\omega\omega}$ are cumulations:

$$\Pr(R_{1\omega}) = \iint_{R_{1\omega}} f_\ell(\ell_1,\ell_2)\,d\ell_1 d\ell_2 = \left[1 - \left(\frac{p_1+\omega}{\theta_1}\right)^{n_1}\right]\left(\frac{p_2+\omega}{\theta_2}\right)^{n_2},$$

$$\Pr(R_{\omega 2}) = \iint_{R_{\omega 2}} f_\ell(\ell_1,\ell_2)\,d\ell_1 d\ell_2 = \left[1 - \left(\frac{p_2+\omega}{\theta_2}\right)^{n_2}\right]\left(\frac{p_1+\omega}{\theta_1}\right)^{n_1},$$

$$\Pr(R_{\omega\omega}) = \iint_{R_{\omega\omega}} f_\ell(\ell_1,\ell_2)\,d\ell_1 d\ell_2 = \left(\frac{p_1+\omega}{\theta_1}\right)^{n_1}\left(\frac{p_2+\omega}{\theta_2}\right)^{n_2}.$$

The last equation is the probability that the outside good is purchased, and is, hence, the share function for that good. Probabilities in R_{12} are:

$$\Pr(\ell_1 > \ell_2 \subset R_{12}) = \left(\frac{\theta_1}{\theta_2}\right)^{n_2} \sum_{j=1}^{n_1} (-1)^{j-1} \frac{{}_{n_1}C_j}{{}_{n_2+j}C_j} \left[\left(1 - \frac{\pi}{\theta_1}\right)^{n_2+j} - \left(\frac{p_2+\omega}{\theta_1}\right)^{n_2+j}\right]$$
$$- \frac{n_2}{n_1+1}\left(\frac{\theta_1}{\theta_2}\right)^{n_2} \sum_{j=1}^{n_1} (-1)^{j-1} \frac{{}_{n_1+1}C_j}{{}_{n_2-1+j}C_j} \left(\frac{p_2+\omega}{\theta_1}\right)^{n_2-1+j} \left[1 - \left(\frac{p_1+\omega}{\theta_1}\right)^{n_1+1+j}\right],$$

for $p_1 - \theta_1 > p_2 - \theta_2$, and

$$\Pr(\ell_1 \le \ell_2 \subset R_{12}) = \left(\frac{\theta_2}{\theta_1}\right)^{n_1} \sum_{j=1}^{n_2} (-1)^{j-1} \frac{{}_{n_2}C_j}{{}_{n_1+j}C_j} \left[\left(1 + \frac{\pi}{\theta_2}\right)^{n_1+j} - \left(\frac{p_1 + \omega}{\theta_2}\right)^{n_1+j} \right]$$

$$- \frac{n_1}{n_2+1} \left(\frac{\theta_2}{\theta_1}\right)^{n_1} \sum_{j=1}^{n_2} (-1)^{j-1} \frac{{}_{n_2+1}C_j}{{}_{n_1-1+j}C_j} \left(\frac{p_1 + \omega}{\theta_2}\right)^{n_1-1+j} \left[1 - \left(\frac{p_2 + \omega}{\theta_2}\right)^{n_2+1+j} \right],$$

for $p_1 - \theta_1 \le p_2 - \theta_2$. Substituting the probabilities into equations (*) and (**) produces the main result of Proposition 7. ■